A topological approach to the problem of emergence in complex systems

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Abstract

Emergent patterns in complex systems are related to many intriguing phenomena in modern science and philosophy. Several conceptions such as weak, strong and robust emergence have been proposed to emphasize different epistemological and ontological aspects of the problem. One of the most important concerns is whether emergence is an intrinsic property of the reality we observe, or it is rather a consequence of epistemological limitations. To elucidate this question, we propose a novel approximation through constructive topology, a framework that allow us to map the space of observed objects (ontology) with the knowledge subject conceptual apparatus (epistemology). Focusing in a particular type of emergent processes, namely those accessible through experiments and from which we have still no clue on the mechanistic processes yielding its formation, we analyse how a knowledge subject would build a conceptual explanatory framework, what we will call an arithmomorphic scheme. Working on these systems, we identify concept disjunction as a critical logical operation needed to identify the system's constraints. Next, focusing on a three-bits synthetic system, we show how the number and scope of the constraints hinder the development of an arithmomorphic scheme. Interestingly, we observe that our framework is unable to identify global constraints, clearly linking the epistemological limits of the framework with an ontological feature of the system. This allows us to propose a definition of emergence strength which we make compatible with the scientific method through the active intervention of the observer on the system, following the spirit of Granger causality. We think that this definition reconciles previous attempts to classify emergent processes, at least for the specific kind we discuss here. The paper finishes discussing the relevance of global constraints in biological systems, understood as a downward causal influence exerted by natural selection. In summary, we think that our approach provides a meeting point for previous efforts on this topic, and we expect that it will stimulate further research in both scientific and philosophical communities.

1 Introduction

"What urges you on and arouses your, you wisest of men, do you call it "will to truth"? Will to the conceivability of all being: that is what I call your will! You first want to make all being conceivable: for, with a healthy mistrust, you doubt whether it is in fact conceivable. But it must bend and accommodate itself to you! Thus will your will have it. It must become smooth and subject to the mind as the mind's mirror and reflection."

Friedrich Nietzsche [1]

Scientific modelling is probably one of the best examples of a human activity fitting the words of Zarathustra: it requires the generation of conceptual representations for processes which depend, many times, on uncomfortable features such as measurement inaccuracy, constituents interdependence or dynamics. We attempt to incorporate these representations within a mathematical or computational framework, which is nothing but a comfortable place where we reaffirm our confidence in the acquired knowledge. Building a formal framework provides a favourable environment for reaching new analytical and computational results, thus accelerating the outcome of new predictions that can be firmly settled within the scientific knowledge after hypothesis testing.

One of the most interesting challenges in scientific modelling relies on complex systems, which are systems composed by a large number of entities driven by non-linear interactions between their components and with the environment. In particular, complex systems may lead to the observation of one of the most controversial phenomena in modern sciences: emergent behaviours. A good intuition for emergence arises when we observe in a complex system that it "begins to exhibit genuinely novel properties that [at a first sight][1]are irreducible to, and neither predictable nor explainable in terms of, the properties of their constituents"[5]. Among this kind of collective behaviours we find phenomena such as magnetism, patterns observed in dissipative systems like hurricanes or convection cells or, in biological systems, patterns on animal skins or flocking behaviour. Looking at these examples it seems that the difficulty relies in an apparent discontinuity between emergent properties and their microscopic description. Since a basic tenet in the scientific method is that macroscopic properties are the consequence of the lower level constituents a critical question arises here [6]: how is it possible to obtain a satisfactory conceptual representation of emergent macroscopic behaviours when the definition of emergence apparently implies a discontinuity between the microscopic and the macroscopic representation?

Explaining the origin of this discontinuity has led to the famous controversy between vitalist and reductionists positioning [7, 3]. We are not interested in entering into this debate, because we do not aim to understand which are the mechanisms making that an emergent property arises. We accept that emergent properties exists and that they are the consequence of the interactions of its lower level properties and the environment –thus we do not accept a compositional physicalism but an explanatory physicalism [2] (otherwise obvious in our view). We are rather interested in answering which are the properties that a complex system may have (and, in particular, those exhibiting emergent behaviours) for being more or less accessible to our knowledge applying the scientific method. In this sense, we align with the proposal of de Haan [8], who highlights the necessity of a general epistemological framework in which emergence can be addressed.

An interesting condition for epistemic accessibility was proposed by Bedau when he coined the term *weak emergence* for those emergent processes that are epistemologically accessible only by simulation [9]. The idea is that a simulation would demonstrate the *supervenience* of upper level properties from lower level constituents, even if the mechanistic process leading to the observed pattern is not completely understood, i.e. it is not possible to compress the simulation into a compact set of rules explaining how the outcome is determined. Therefore, it provides an objective definition of emergence based on computational incompressibility, that has been explored by different models such as cellular automata [10, 11] or genetic algorithms [12], approximations that were later called *computational emergence* [13].

Nevertheless, Bedau pointed out as well that, even if there is computational incompressibility, it could be possible to recover from the simulations regularities that may lead to describe the system under compacts laws [9]. Otherwise, the above computational approximations should be understood as non-epistemic, as they would be useless to decipher the principles governing the emergence of a property. Furthermore, Huneman rightly emphasized that this is an important issue to understand the relationship between computational models and processes observed in nature [14] and, for those computational models depicting regularities in their global behaviour, coined the term *robustly emergent*.

There is a last notion of emergence we would like to discuss that has been considered fundamental –as opposed to epistemological–, which is called *strong emergence* [15]. In Physics it is accepted that knowing the positions and velocities of particles is sufficient to determine the pairwise interactions. This assumption is frequently found in Physics-inspired models of collective behaviour, where individual motion results from averaging responses to each neighbour considered separately. Nevertheless, Bar-Yam reasoned that this assertion would not hold if the system is embedded in responsive media –such as the motions of impurities embedded in a solid–, or in any process where global optimization (instead of local) is involved. In this way, if there is in the system a constraint acting on every component simultaneously and it is strong enough –i.e. it is a global constraint–, it is not possible to determine the state of the system considering only pairwise interactions. In some sense, the parts are determined *downwards* from the state of the whole, being the consciousness the most paradigmatic example suggested for strong emergence.

In this paper, we are interested in understanding which are the conditions that a natural system depicting an

[1]The statement in brackets is ours because, although we think that this definition provides a good intuition on emergent processes, we agree with Mitchell in that it is simplistic [2]. We rather adhere to the view of Francis Crick when he explains that "The scientific meaning of emergent, or at least the one I use, assumes that, while the whole may not be the simple sum of its separate parts, its behaviour can, at least in principle, be understood from the nature and behaviour of its parts plus the knowledge of how all these parts interact" (reproduced from [3] referring to [4] p.11)

emergent property may have for fitting the notion of robust emergence. However, we will shift the attention from computational models to focus on the analysis of experimental data. We will follow the view in which emergence is considered a "relation between descriptions of models of natural systems and not between properties of an objective reality in itself" [16] although, as we will see immediately, we will not renounce to talk about ontology. In particular, we aim to understand the relation between macroscopic descriptions of emergent behaviours and their microscopic explanatory models. Starting from experimentally characterized microscopic states associated to a macroscopic emergent observation, we want to understand which kind of regularities found in these microstates are more difficult to compress and why. We believe that this is a necessary *a priori* step for any computational approach aiming to model the observed process, and thus scientifically relevant notions of emergence should be derived from this process rather than from the computational models themselves.

An immediate risk in this endeavour is that we must select a framework to work with and, by doing this, any result may depend on the framework selected. To circumvent this difficulty, we will apply a framework founded in logic whose limits are thus clearly established. In particular, we will analyse with logical principles the map between the conceptual setting build by an observer following the scientific method and the objects observed. As a consequence, we will be able to clearly see why and how this limits are surpassed, and this map will help us to recover the ontological meaning of these limitations.

The article is articulated as follows. In Section 2 we introduce the formalism. We carefully provide a definition of system, introducing next the different mathematical tools needed and finally providing a topological notion of conceptual vagueness. We will show here that well known difficulties discussed around the concept of emergence [7, 17, 18, 8, 9, 19], can be understood through this notion of vagueness. Thus, our effort in the application of a novel formalism becomes justified because we find a more expressive picture of the epistemological problems we face when dealing with complex systems and their ontological origins. We analyse in Section 3 the macroscopic and microscopic descriptions, showing how the macroscopic descriptions are more prone to generate vague concepts while the (bottom-up) microscopic description is used to build explanatory frameworks. It is in Section 4 where we will show how the knowledge subject proceeds to build such a framework, and we discuss how concept disjunction helps us in the identification of the system's constraints. Then we go into three synthetic systems in which, applying the new formalism, we identify which are the properties that make the system more or less epistemologically accessible. We finish in section 5 discussing the limitations of our approach, and how to elucidate which is the complexity of the constraints underlying the system under analysis despite of these limitations, what lead us to propose a definition of emergence strength.

2 A topological description of the phase space induced by measurable properties

In the following sections we introduce a novel application of topological notions, whose novelty relies on its ability to formally describe epistemological questions that are hardly addressed by other approximations. A nice introduction for computational scientists to the generalization of the approach presented here, called formal topology, can be found in [20], and a relevant application to the epistemological determination of what should be understood as a vague concept is found in [21].

2.1 System definition

We start proposing a glossary of terms concerning the system definition, some of them close to those proposed by Ryan in [17]. We will call (object of) observation o_i, to a set of basic magnitudes associated to a given entity. Each of this magnitudes is a function f_M of the Cartesian product of a collection of M sets –where at least one of them is determined by an experimental measurement–, into real numbers $\mathbb{R}$, i.e. $f_M : A \times B \times ... \times M \to \mathbb{R}$.

The non measurable sets may refer to a set of measurement units (grams, meters,...) to a set of reference frameworks, or any other set necessary to determine the final magnitude. For simplicity, we will consider that any variation in the magnitudes is a consequence of a variation in the outcome of a measurement and thus, in the following, we will not distinguish between magnitude and measurement when objects of observation are discussed.

In this way, we will consider that our system is characterized by a bunch of M quantitative and/or qualitative (i.e. binary) magnitudes $X = \{x_k, k = 1, .., M\}$. Given that we are interested in complex systems, we will consider that our system consists of a large number of entities, that we denote with N. We will call *scope* to this selection of objects whose size, N, implicitly determines the spatio-temporal boundaries of the system. Determining the scope is already a difficult task for large dynamical systems. These difficulties arise, on the one hand, from the identification of these entities because, when the number is large, a complete characterization may be unfeasible. On the other hand, it will be also difficult to define the separation between system and environment, as this separation cannot be achieved many times using strictly objective arguments [22, 23]. We will discuss these questions in more detail below.

The variables selected X are intended to be sufficient to answer the questions addressed in the research. For simplicity in the exposition, we start considering an ideal scenario in which all these variables can be quantified for any entity within the system, leading to $N \times M$ specific values. This assumption will not affect our conclusions, as we can assign a vanishing value to any variable from which the associated magnitude is not observed for one or several entities. We will propose a procedure to relax this assumption in section 5.2 to discuss how the complexity of a system can be explored following the scientific method.

Every variable x_k has a *resolution* $r_k = f : x_k \to \mathbb{R}$, which is the finest interval of variation that we set for that variable, and it is established from different arguments. For instance, the resolution may be limited by the intrinsic error in the measurement, which would be an ontological limitation. Another possibility arises when the expected influence of a given variable on the system's description is small for a given shift in the value, and a coarser discretization is then justified (an epistemological limitation). Calling I_k to the domain of the function $f : x_k \to \mathbb{R}$, the number of possible values considered for the variable x_k will be $\zeta_k = I_k/r_k$. We will call resolution R of the system to the finest variation that allow us to distinguish two states of the system, $R = \max_k(\{\zeta_k\})$ $(k = 1, .., M)$.

This choice of variables together with the set of viable values will be called the *focus* F of the knowledge subject, upper bounded by $F \sim M \times R$. We finally call the *scale* to the set of specific values $\{N, M, R\}$. A factor multiplying any of these values represents a change in the scale of the scope (if we modify N), or the focus (if we modify M or R). Note that, following this definition, the scale is an ontological attribute as determined by N, but it also depends on the epistemological attributes determined by M and R. Therefore, the breadth of the focus is very much influenced by epistemological choices. Interestingly, it has been suggested that emergent behaviours (theories) are the consequence of a change in the scope [17] (and not in the focus [19]).

2.2 Measurable properties, concepts and their extension.

Let us start introducing some definitions, most of them already provided and justified in [21], that we recover here for completeness. For the sake of simplicity we will start considering that our objects of observation $o \in O$ are the components of a complex system at a given time, i.e. we focus on a single microstate μ with N components described by M variables with resolution R. Each of these components is what we consider for the moment an object of observation. We will move later towards a description where each object of observation is a microstate, becoming the whole space of objects the observed phase space. All the definitions considered in the following for a single microstate can be extended for other objects with a different scale.

Definition: We call a *basic concept* or *characteristic* $c_a = x_k^*$ to the specific value x_k^* of a variable x_k, out of the ζ_k possible values, measured over an object of observation o. In this way, if we consider two different measurements of our variables for the same entity, each of them will constitute a different object of observation.

Definition: We call *focus* F to the whole set of characteristics considered by the observer: $F = \{x_k^l;\ k = 1, .., M;\ l = 1, .., \zeta_k\} \equiv \{c_a;\ a = 1, .., \tilde{M}\}$, with $\tilde{M} = M \times \sum_k \zeta_k$. We make explicit here the discrete nature of the conceptual setting and the relation between resolution and focus, which achieves a suitable description in terms of characteristics, in turn leading to the definition of concepts. Note that discreteness here is not an arbitrary choice, because we are working with experimental values that have associated an experimental error, making the measurements essentially discrete. The transition into continuous descriptions is a posterior formal abstraction made during the modelling process.

Definition: We call a *concept* ν to any non-empty finite subset of F: $\nu = \{c_1, ..., c_P\}$, with $P \leq \tilde{M}$. We defined the intension of concepts. Given that a concept may contain a single characteristic, $\nu = \{c\}$, any characteristic can be considered a concept as well. The distinction between concept and characteristic will be needed in certain circumstances to indicate a qualitative difference that makes a concept a more elaborated entity. For instance, a characteristic may be the outcome of a measurement and a concept could be the result of that measurement together with its units. Therefore, the term characteristic will be used to talk about elementary concepts (such as single measurements outcomes) but, apart from its utility in making precise the definitions, this distinction is not necessary for our purposes and, for the sake of a simplified exposition, both terms will be used interchangeably.

2.3 Binary operations

From the previous definitions it is immediate to propose binary operations to build new concepts.

Definition: (Conjunction of concepts). Let $\nu_1 = \{c_1, ..., c_P\}$ and $\nu_2 = \{d_1, ..., d_Q\}$ be two concepts. Then, the conjunction of $v_1$and v_2 is the concept

$$\nu_1 \wedge \nu_2 = \{c_1, ..., c_P, d_1, ..., d_Q\} \tag{1}$$

The conjunction of concepts is, in turn, a concept which consists on the set of all the characteristics contained in both concepts. Alternatively, we may want to extract, given two concepts, the common characteristics they share:

Definition: (Disjunction of concepts). Let $\nu_1 = \{c_1, ..., c_P, b_1, ..., b_L\}$ and $\nu_2 = \{d_1, ..., d_Q, b_1, ..., b_L\}$ be two concepts. Then the disjunction of $v_1$and v_2 is the concept

$$\nu_1 \vee \nu_2 = \{b_1, ..., b_L\} \tag{2}$$

The disjunction of concepts leads to a concept containing the set of all characteristics common to both concepts. Note that the set of concepts we consider determines a partition of the focus and it will induce, in turn, a partition in the set of objects of observation. In other words, understanding the relationship between these partitions requires to determine a constitutive relationship between any single characteristic belonging to the focus F and the set of objects O. The following constitutive relationship will express that the objects become cognitively significant by means of the characteristics measured and, in turn, by the concepts we build from them:

Definition: (Constitution relation). Let F be the focus over a set O of objects. Given $o \in O$ and $\nu \in F$, we introduce a binary relation, $\Vdash$, that we call *constitution relation*, such that by $o \Vdash \nu$ we mean that ν is one of the concepts constituting o.

With the constitution relation we determine how the objects of observation are expressed via the conceptual apparatus of the knowing subject. In addition, we would like to know which objects are constituted by a given concept:

Definition: (Extension of a concept). Let $\nu \in F$ be a concept. Then the extension Ext of ν is the subset of objects of O constituted by ν, that is

$$\mathrm{Ext}(\nu) = \{o \in O \mid o \Vdash \nu\} \tag{3}$$

We note here that an immediate consequence of Eq. 3 is that any object of observation has necessarily associated a concept, i.e. it is just cognitively accessible by means of the conceptual apparatus of the knowing subject. This assertion, if accepted in general, leads to a Kantian epistemological positioning [21]. In our case, it is a consequence of the fact that our objects of observation are built from measurements of a reproducible experimental setting, and hence it is true by construction. Nevertheless, the opposite is not true as we may deal with concepts for which no object is observed, i.e. $\mathrm{Ext}(\nu) = \emptyset$. These concepts are considered for instance if we have *a priori* expectations of the viable values of the system[2].

Finally, we aim to know what is the extension of a subset U of concepts $U = \{\{\nu_1\}, ..., \{\nu_L\}\}$.

[2]For instance, we know that a group of birds can fly following any direction in the space even if we systematically observe that they follow a single direction. From the point of view of the scientific method, concepts build from *a priori* expectations are very important, as they may be used to propose null hypothesis which, in general, can be formulated as H_0 : "ν observed". It is when we reject the hypothesis through experiments when we acquire a scientific knowledge of the process analysed, i.e. that ν is not observed.

Lemma: Let U be a subset of the set F of concepts. Then, the extension of U is defined by setting

$$\mathrm{Ext}(U) = \bigcup_{\nu \in U} \mathrm{Ext}(\nu) \tag{4}$$

Hence, if we consider two concepts c_1 and c_2, we should not confuse a concept built by conjunction of concepts $c = c_1 \wedge c_2$ with the subset of concepts $C = \{\{c_1\}, \{c_2\}\}$. In the former case, we look for objects containing both concepts and, thus, the number of such objects is smaller or equal than the number of objects described by c_1 or c_2. On the other hand, the subset C extends over objects containing *any* of the concepts, being its extension the union of the extension of both concepts. In this paper, we will analyse sets of objects and we will be interested not only in finding a description that would allow us to "talk" about them but, in addition, we will look for minimal descriptions, i.e. descriptions containing the lowest number of concepts. We will show below that disjunction is the basic logical operation we need to obtain such representations.

2.4 Topology and vagueness

We introduce now some more definitions and a theorem, which represent the basis of the topological approach [21].

Theorem 1: If the map Ext satisfies the extension condition, then the family $\{\mathrm{Ext}(U) | U \subseteq F\}$ is a topology over the set O, where U is a subset of concepts of the focus F.
This map is central in our arguments. The basic characteristics are defined in terms of measurements over specific objects, and thus the extension provides a map between these characteristics and the sets of objects. If we call power set $\wp$ to the set containing all the possible partitions in which a given set (in our case O) can be divided, a topology will be some subset of the power set containing a collection of sets called *open sets* –which include the empty set and the whole set–, and verifying: 1) the arbitrary union of open sets is another open set in the topology; 2) The binary intersection of open sets is also another open set in the topology. Therefore, a topology is a subset of $\wp$ which is *closed* under arbitrary union and binary intersection of the open sets it contains.

What we are expressing is that, once we have a conceptual setting built from measurements, the extension function induces a partition in the set of objects, and this partition fulfils the conditions for being a topology. In this way ,we can take advantage of the topological notions of open and closed sets. Justification for the following definitions can be found in [24, 21]

Definition: (Open set) Let A be a subset of the set O of objects. Then A is an open set if it coincides with its interior $\mathrm{Int}(A)$, where

$$\mathrm{Int}(A) = \{o \in O \mid (\exists \nu \in F)\ o \Vdash \nu \ \&\ ext(\nu) \subseteq A\}. \tag{5}$$

Definition: (Closed set) Let A be a subset of the set O of objects. Then A is a closed set if it coincides with its closure $\mathrm{Cl}(A)$, where

$$\mathrm{Cl}(A) = \{o \in O \mid (\forall \nu \in F)\ o \Vdash \nu \Rightarrow (\exists o \in O)\ o \in ext(\nu)\ \&\ o \in A\}. \tag{6}$$

Definition: (Border) Let A be a subset of the set O of objects. Then the border $\mathrm{Bd}(A)$ of A is the set

$$\mathrm{Bd}(A) = \mathrm{Cl}(A) \cap \bar{A} \tag{7}$$

where $\bar{A}$ stands for the complement of the set A with respect to O.

Definition: (Vagueness) Let ν be any concept and U be any set of concepts. Then ν is a vague concept if $\mathrm{Bd}(\mathrm{Ext}(\nu))$ is non-empty, and U is a vague set of concepts if $\mathrm{Bd}(\mathrm{Ext}(U))$ is non-empty.
As we anticipated, this definition of vagueness will help us to understand the origin of ambiguities associated to the definition of emergence.

3 Vagueness and descriptions of the system

3.1 Microscopic and macroscopic descriptions

Following the definitions introduced we aim now to differentiate two types of variables providing a description of the system at different scales: microscopic and macroscopic. Note that with microscopic we do not mean "atomistic", we just talk about a significantly shorter spatio-temporal scale of observation of the system. A particular feature of the interplay between both scales is that, when a macroscopic property is observed during the dynamical evolution of a system, even if the microscopic variables are continuously changing, the macroscopic variables remain invariant.

In the following, we will call microstate μ to a vector containing, at a given time, the values of a set of variables $\{x_k\}$ that fully determines the state of the microscopic objects, i.e. $\mu = \{x_k^*\}$ where x_k^* stands for a particular value of the variable x_k. Therefore, the basic objects of observation we are considering now are the microstates $o \equiv \mu$. When a coarse graining of the microstates at the spatial, temporal or both dimensions is performed, it may be possible to determine macroscopic variables y_k describing the state of the system in the new (coarser) scale. We will call macrostates to the objects described at the macro scale $o \equiv \xi$. In some cases, the macroscopic variables y_k can be obtained applying a surjective map f over the microscopic variables $f(x_k) \to y_k$. For instance, if we deal with an incomplete (statistical) microscopic description of an ensemble $P(\mu)$, we can obtain a coarse determination of a macroscopic variable y_k averaging the correspondent microscopic variable x_k weighted by the statistical probability of the microstates over the ensemble $\langle x_k(\mu)P(\mu)\rangle$. But in many other situations it is not possible to find out such mapping, and we argue here that this fact underlies the problems surrounding the study of emergent properties: we observe a macroscopic property such as the collective behaviour of many interacting elements, and it does not seem possible to explain it from lower levels of description (for instance from the properties of the entities themselves).

It is important to underlie that macrostates and microstates definitions are relative to the scale of observation and they may change if we move from one scale of description to another. Consider a system described within a certain temporal scale by a set of microstates $\{\mu_i\}$ which are associated to the observation of a single macrostate ξ. Assume now that the system evolves under a sufficiently long path such that we observe different macrostates and we store T snapshots of this dynamics, leading to an ensemble of macrostates $\{\xi_u\}_{u=1}^{T}$. It is possible to consider that each of these macrostates is now a microstate $\hat{\mu}$ for a new system with a larger scope and lower resolution $\xi_u \to \hat{\mu}_i$. Given that the scope of a macrostate will be always larger than that of a microstate ($N_\xi \geq N_\mu$), whereas it occurs the opposite with the resolution ($R_\xi \leq R_\mu$), in this exercise we have increased the scope and reduced the resolution. This is the reason why, larger is the scale, more difficult is to build a bottom-up explanatory framework. This movement along different scales will be very relevant when evolutionary systems are considered, given that we will need to distinguish at least two spatial and temporal scales. For instance, such a change is needed when moving from the ecological analysis of few individuals to the evolutionary analysis of entire populations.

Note as well that this change in the scale requires an effort to reduce the system description, but this kind of reduction has been performed from the very first step: for the definition of scope, we have neglected entities; for the focus, we have neglected variables and probably restricted their viable values assuming a lower resolution. Furthermore, any map between microstates and macrostates again considers a reduction in the information provided by the microstates. In general, for both very broad or very detailed questions the technical complexity increases and a reduction in the description is unavoidable, and it is important to remark that this exercise does not mean that the approach is reductionist. Reductionism should be considered an epistemological attitude where it is accepted that any macroscopic description is a simple extrapolation of the properties of the microscopic description [25]. Instead, we accept that in complex systems there are discontinuities between the different levels of description and that, for each new level, new properties may arise. We are interested here in investigating which are the minimum conditions to say that a microscopic description is a valid representation of an emergent macroscopic observation.

3.2 Vagueness in the macroscopic description and dialectic concepts

In this section, we aim to justify with a simple example a plausible origin for the controversies around emergent properties. Importantly, we will not provide any *a priori* definition of emergent property. For the moment, we just assume that there are processes that are sufficiently surprising for any observer as to qualify them as emergent. We argue that, given that the macroscopic scale is less refined in terms of number of variables and values, it is more prone to generate dialectic concepts. Therefore, we expect that it is in the macroscopic scale where emergent properties are first identified.

Let's start considering a simple example where we investigate a complex system for which two different concepts can be built macroscopically. The first concept $\hat{c}_\Omega$ describes the performance of the complex system when it is able to visit every possible viable values of the focus, and thus the subscript Ω indicates that it describes a behaviour found for any microstate of the phase space, i.e. it is not constrained. Remember that we use the *hat* over the concept to denote that it is a macroscopic concept. For concreteness, let us assume that the concept is $\hat{c}_\Omega =$ "groups of birds flying". Now consider that, from time to time, we observe that these birds depict a swarming (or flocking) behaviour and we coin an specific concept for this $\hat{c}_{\mathrm{E}} =$ "flocking", where the subscript E stands for "emergent", just because we find the behaviour novel and appealing. If we call $\{o_{\mathrm{E}}\}$ to the observations where we appreciate a flocking behaviour and $\{o_{\mathrm{NE}}\}$ to those where we do not, the conceptual setting describing the system will be:

$$\begin{aligned} \mathrm{Ext}(\hat{c}_0) &= \quad \emptyset \\ \mathrm{Ext}(\hat{c}_{\mathrm{E}}) &= \quad \{o_{\mathrm{E}}\} \\ \mathrm{Ext}(\hat{c}_\Omega) &= \quad \{o_{\mathrm{E}}, o_{\mathrm{NE}}\}. \end{aligned}$$

where $\hat{c}_0$ is the concept describing the null observation of the system. According with the previous definitions, $\mathrm{Bd}(\hat{c}_{\mathrm{E}}) = \{o_{\mathrm{NE}}\}$ and we identify $\hat{c}_{\mathrm{E}}$ as a vague concept. The fact that the border of the concept is the set of observations *not* depicting flocking $\{o_{\mathrm{NE}}\}$ behaviour means that we are still not good at explaining why these observations do not belong to the set of observations that do show flocking behaviour $\{o_{\mathrm{E}}\}$. As a consequence, the set $\{o_{\mathrm{NE}}\}$ cannot be *safely separated* from $\{o_{\mathrm{E}}\}$. The reason is that the concept $\hat{c}_{\mathrm{E}}$ is still not informative of what "flocking" means and, as a consequence, we cannot use it to say what does not mean, what would help us in separating both sets (i.e. we cannot properly build a concept $\hat{c}_{\mathrm{NE}} = \neg\hat{c}_{\mathrm{E}}$, where $\neg$ is the NOR operator). This is indeed an important question to work under the scientific method, because we need to establish hypothesis of the type H_0 : "d observed" which, upon rejection, lead to the concept $c = \neg \mathrm{d}$ (d is not observed).

David Bohm pointed out that, in the earlier stages of any science, the interest is focused on "the basic qualities and properties that define the mode of being of the things treated in that science" [26], being tasks such as comparative analysis and classification the cornerstones in its earlier development. We argue, that it is in a descriptive (exploratory) context in which the larger number of vague concepts arise. Following the classification of concepts proposed by Georgescu-Roegen [27] we will refer to concepts containing any source of vagueness as dialectic. To this type belong breadth concepts [28], those related with dynamical properties like stability [29], the difference between organism and machine [30] or the exact intension of function, autopoiesis and complexity [7]. These are classical examples of dialectic concepts and it is remarkable the potential these concepts have for generating debate, which should be considered an asset [31].

To continue with Bohm's view on scientific evolution, it is just after a sufficient exploitation of the dialectic knowledge when we will find a growing interest on "processes in which things have become what they are, starting out from what they once were, and in which they continue to change and to become something else in the future" [26]. This knowledge, following again the classification of concepts proposed by Georgescu-Roegen, is built on arithmomorphic concepts: "[arithmomorphic concepts] conserve a differentiate individuality identical in all aspects to that of a natural number within the sequence of natural numbers"[27]. Arithmomorphic concepts are suitable for formal reasoning and quantitative treatment, and we argue that emergent processes are dialectic macroscopically, and that intense research is developed around them upon a successful microscopic arithmomorphic scheme is built.

3.3 Microscopic description and arithmomorphic scheme

Microscopically, when $\{o_{\mathrm{NE}}\}$ is observed the system depicts a dynamical evolution where constraints are absent. Constraints here should be simply understood as limitations in the viable values of the variables we handle [32]. Any system is constrained in some extent. But there are some constraints that belong to the definition of the system itself, that we will call *intrinsic*, and others that depend on particular conditions to be observed, that we will call *facultative*. Think in the structural differences between a protein, and a heteropolymer whose sequence is the result of randomly shuffling the protein sequence. Although both chains of amino-acids depict the same number of intrinsic constraints (those derived from the existence of peptidic bonds), a protein structure requires three additional constraints levels: the first is needed for being kinetically foldable in a biologically relevant time, the second for making the fold thermodynamically stable under physiological conditions, and the third for performing its specific function (metal-binding, phosphorilation, etc.). Both chains have the same amino-acids but the evolutionary process has selected for an specific order in the sequence that generates the constraints needed to make possible that the emergent property (the protein function) arises. Quantitatively, the probability that these constraints appear by chance is quite low: the number of possible heteropolymers of length N, considering an alphabet of 20 amino-acids, is 20^N and the number of protein structures (of *any* length) deposited at date in the Protein Data Bank is $\sim 1.2 \times 10^5$ (www.pdb.org). If we call $\{\mu_{\mathrm{E}}\}$ the set of microstates where the emergent behaviour is observed, $\{\mu_{\mathrm{NE}}\}$ the region where it is not observed, and Ω the whole phase space, we expect that $\#\{\Omega\} \approx \#\{\mu_{\mathrm{NE}}\} \gg \#\{\mu_{\mathrm{E}}\}$; where $\#\{\cdot\}$ is the cardinality of the set, i.e. the number of elements it contains. Therefore, we expect that the volume of the region of the phase space were an emergent property arises to be much smaller than the whole phase space. This is probably why unpredictability or surprise are attributes frequently used for emergent properties.

Recovering the example of Sec. 3.2 we know that the following map between the macroscopic description and the underlying processes exists:

$$
\begin{aligned}
\mathrm{Ext}(\hat{c}_0) &= \emptyset \\
\mathrm{Ext}(\hat{c}_{\mathrm{E}}) &= \{\mu_{\mathrm{E}}\} \\
\mathrm{Ext}(\hat{c}_{\Omega}) &= \{\{\mu_{\mathrm{NE}}\}, \{\mu_{\mathrm{E}}\}\}.
\end{aligned}
$$

Again, the concept $\hat{c}_{\mathrm{E}}$ determines an open whose closure is the whole phase space $\mathrm{Cl}(\mathrm{Ext}(\hat{c}_{\mathrm{E}}) = \{\mu_{\mathrm{E}}\}) = \{\{\mu_{\mathrm{NE}}\}, \{\mu_{\mathrm{E}}\}\}$. The intersection between the closure and this open determines its border $\mathrm{Bd}(\mathrm{Ext}(\hat{c}_{\mathrm{E}}) = \{\mu_{\mathrm{E}}\}) = \{\mu_{\mathrm{NE}}\}$ which is non-empty which, as we have seen, means that the concept $\hat{c}_{\mathrm{E}}$ is a vague concept. Getting into the microscopic description, vagueness will vanish if we build a topology such as:

$$
\begin{aligned}
\mathrm{Ext}(c_0) &= \emptyset \\
\mathrm{Ext}(c_{\mathrm{E}}) &= \{\mu_{\mathrm{E}}\} \\
\mathrm{Ext}(c_{\mathrm{NE}}) &= \{\mu_{\mathrm{NE}}\} \\
\mathrm{Ext}(c_{\Omega}) &= \{\{\mu_{\mathrm{NE}}\}, \{\mu_{\mathrm{E}}\}\}
\end{aligned}
$$

where $c_{\mathrm{NE}} = \neg c_{\mathrm{E}}$. In the examples we develop in the next sections, we will call generically to the positive (null) observation with the concepts c (d). For all the systems analyzed, we will assume that some microstates correspond to a macroscopic emergent observation, and vagueness removal will be achieved when we are able to find a subset of concepts U such that $\mathrm{Ext}(U_{\mathrm{NE}}) = \{\mu_{\mathrm{NE}}\}$, which means that, when only the emergent process is observed $\mathrm{Ext}(U_{\mathrm{NE}}) = \emptyset$, we reject the hypothesis H_0 : "U_{NE} is observed". Therefore, in the following section we aim to investigate how we can build a microscopic arithmomorphic description looking for microscopic concepts that allow us to differentiate between the sets of microstates $\{\mu_{\mathrm{E}}\}$ and $\{\mu_{\mathrm{NE}}\}$.

4 A synthetic approximation to emergent properties

4.1 Traceability and compact descriptions

With the above considerations, we propose two formal definitions that will be helpful to understand our rational behind the further development of the paper. We first define what is considered a novel macroscopic property, which identification is typically the starting point of any research.

Definition: (Novel macroscopic property). We will say that an observed macroscopic property is a novel property if it is observed only in the presence of certain facultative constraints limiting the viable values of the system.

Therefore, given that the phase space of the system Ω is restricted to a smaller observed region $\Omega^{\mathrm{O}} \subset \Omega$, what we say is that there exists a macroscopic concept $\hat{c}$ such that $\mathrm{Ext}(\hat{c}) = \Omega^{\mathrm{O}}$, and we would like to explore this region both in terms of macroscopic $\{\hat{c}\}$ and microscopic $\{c\}$ concepts. We now introduce a condition that allow us to consider that a macroscopic description is in correspondence with a microscopic description.

Definition: (Traceability). Given a novel macroscopic property $\hat{c}$ and the observed phase space Ω^{O} associated to that property, i.e. $\mathrm{Ext}(\hat{c}) = \Omega^{\mathrm{O}}$, we will say that the macroscopic description obtained is traceable if we find an appropriate function or algorithm applied on microscopic

properties $f : \{c\} \rightarrow q$ such that the new concept q derived *compactly* describes the ensemble of microstates, i.e. $\mathrm{Ext}(q) = \mathrm{Ext}(\hat{c}) = \Omega^{\mathrm{O}}$**.**
This definition paves the way for quantifying the correspondence between both descriptions within the framework proposed. Note that it does not require to be able to relate macroscopic and microscopic properties, but only to establish a correspondence between microscopic and macroscopic variables describing the same region of the observed phase space Ω^{O}. Therefore, traceability can be seen as a rather minimal epistemological condition, because it is what allow us to talk about emergent properties circumventing any epistemological discontinuity between both descriptions. Still, we need to clarify what is understood by compact description.

Definition: (Compact description) We say that a set of distinguishable microstates $\{\mu\}$, namely a set in which every microstate μ is completely described through the microscopic set of concepts F, is compactly described by a concept q, if $\mathrm{Ext}(q) = \{\mu\}$ **and the number of concepts needed to build q is strictly smaller than F.** In the next sections we show with different examples how can be achieved a compact description in order to say that a novel macroscopic property is traceable.

4.2 Identification of constraints: focusing on disjunction

In this section, we would like to investigate if there is any general method to reach a compact description of the ensemble of microstates. As we said, when an emergent property is observed, the system is constrained to a certain region of the phase space. This means that there is a breaking of symmetry, namely the probability distribution for values of the different variables depart from the distribution observed when the system is free of constraints, thus losing ergodicity [33] (p. 186). Therefore, the existence of facultative external or internal constraints limit the behaviour of the system and, as we will attempt to clarify, a necessary condition for determining a microscopic property associated to every microstate visited requires the determination of the existing constraints. We will see that the nature of the different constraints acting on the system determine its epistemological accessibility and, thus, our ability to reach a satisfactory explanation of emergent behaviours.

Firstly, we show how can be obtained the extension of those concepts built through binary operations over sets of concepts. When we obtain a new concept τ via conjunction, for instance $\tau = \nu_1 \wedge \nu_2$, the extension of the new concept will be the intersection of the sets of objects associated to each of the starting concepts $\mathrm{Ext}(\tau) = \mathrm{Ext}(\nu_1) \cap \mathrm{Ext}(\nu_2)$. Aiming to fully identify a single object requires to determine a sufficiently large number of concepts in order to sharply separate it from the other objects, being conjunction the basic operation that permits to reach more precise descriptions.

Let us take as an example the description of a set of proteins $\{o_\alpha\}$ provided by the sequence of their amino-acid composition, which is embedded within an evolutionary phase space. Each amino-acid molecule in the protein is a component of the system which is described by its position in the sequence and by a single variable whose specific value consists of one out of the 20 natural amino-acids encoded by DNA. In this way, an example of concept within this description would be something like $\nu_i =$"*cysteine in position i*" –which, in turn, is built by conjunction of the more basic characteristics "*cysteine*" and "*i*"–. A protein sequence o_α will be subsequently built by conjunction of a set of such a kind of concepts describing the amino-acid observed at each position, i.e $\alpha = (\nu_1 \wedge \nu_2 \wedge ... \wedge \nu_N)$ (see Fig. 1). The sequence becomes uniquely determined under this description, i.e. the extension of the sequence maps exactly one object of observation, namely, the protein under study: $\mathrm{Ext}(\nu_1 \wedge \nu_2 \wedge ... \wedge \nu_N) = \mathrm{Ext}(\alpha) = o_\alpha$. In summary, conjunction underlies bottom-up approximations, where we focus in an accurate description through the compilation of concepts.

Let us now have a look to another kind of concepts λ, which are obtained via disjunction, $\lambda = \nu_1 \vee \nu_2$. Following the equation 2, given that $\nu_1 = \{c_1, ..., c_P, b_1, ..., b_L\}$ and $\nu_2 = \{d_1, ..., d_Q, b_1, ..., b_L\}$, the extension of the concept $\lambda = \{b_1, ..., b_L\}$ will be given by $\mathrm{Ext}(\lambda) = \mathrm{Ext}_{i<j}(b_i \wedge b_j)$ (see Fig. 1). From this definition we note that $\mathrm{Ext}(\{\{\nu_1\}, \{\nu_2\}\}) = \mathrm{Ext}(\nu_1) \cup \mathrm{Ext}(\nu_2) \subseteq \mathrm{Ext}(\nu_1 \vee \nu_2)$. Hence, this is an operation that allow us to find commonalities, which may be extended to objects that are not included in the sets of objects over which the concepts ν_1 and ν_2 are extended. Disjunction stands out as a relevant operation to look for breadth concepts, and it is consistent with the intuition stating that these concepts tend to overtake the boundaries of our starting focus.

4.3 The three bits system

We are already equipped with all the tools necessary for analysing in detail a synthetic example. The toy model we consider consists on a system of three entities whose physical state is described by a single binary

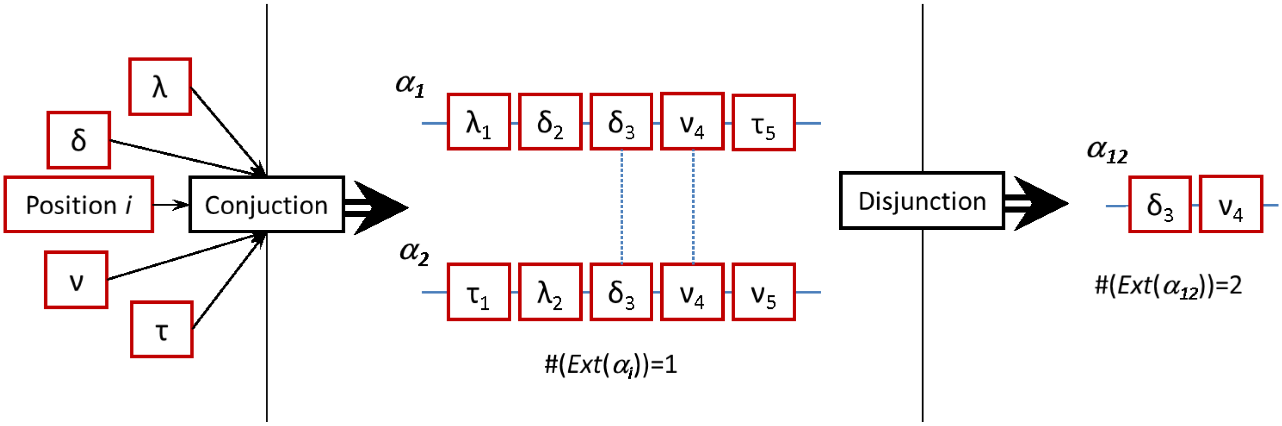

Figure 1: **Illustration of conjunction and disjunction of concepts.** Starting from the knowledge subject's conceptual apparatus (greek letters, left), two sequences α_1 and α_2 are built through conjunction of the basic concepts, being themselves concepts (center). These sequences uniquely determine a single object, for instance a protein sequence, and thus $\#(\mathrm{Ext}(\alpha_i)) = 1$. By comparison of both sequences we observe two common concepts (linked by dotted lines) that we extract through binary disjunction leading to a concept α_{12} (right) containing less basic concepts but whose extension is larger than the original sequences ($\#(\mathrm{Ext}(\alpha_{12})) = 2$) being its scope larger. In the case of proteins it may be understood as a signature of their common ancestry, i.e. of their homology.

variable, i.e. a system modelled with three bits. From an experimental point of view, there may be three distinguishable entities (ranging from a molecule to a population) described with binary variables. We can think in sets of genes that are expressed (not expressed) when the amount of the correspondent protein is above (below) certain threshold, species observed (absent) in certain environmental sample or, from a strictly computational experiment, the attractor of a boolean network. Each measurement performed over these entities will be considered an observation, and each of them may take a value of one or zero. For a system composed by three entities we can observe $2^3 = 8$ microstates $\mu_k = (x_1, x_2, x_3)$ (with $k = 1, .., 8$; and $x_i = \{0, 1\}$; see table 1). As the three entities can be distinguished the focus is

$c_1 =' ON\ at\ object\ 1';\ d_1 =' OFF\ at\ object\ 1';$

$c_2 =' ON\ at\ object\ 2';\ d_2 =' OFF\ at\ object\ 2';$

$c_3 =' ON\ at\ object\ 3';\ d_3 =' OFF\ at\ object\ 3';$

Each microstate is defined in terms of this focus through concepts e_k ($k = 1, ..., 8$) built by conjunction of characteristics. For instance, the microstate $\mu_7 = (1, 1, 0)$ is defined in terms of the basic characteristics as $e_7 = c_1 \wedge c_2 \wedge d_3$, being in turn a concept. The fact that we work with elementary concepts build from measurements in which both the observation or not of certain property can be assessed experimentally, stands on the basis of the construction of a bottom-up explanatory framework. The observation of a "flocking behaviour" cannot be assessed experimentally unless we clearly determine measurable values. A bottom-up characterization in terms, for instance, of the relative angles in the positions between birds allow us to tackle the vagueness associated to the term "flocking". Furthermore, it has been claimed that flocking is not observed in a single microstate [17]. However, once we have characterized flocking with microscopic variables, for example once we derive the pairwise distribution of interactions among neighbours [34], we can also test whether any behaviour, even for a single snapshot in the flight of a group of birds, is consistent with the model derived. This fact already provides the means to build an hypothesis testing experiment where the null hypothesis is that the group of birds do not present flocking behaviour.

We can define for the three-bits system $\binom{N}{n_{\mathrm{ext}}}$ possible combinations of constraints involving n_{ext} variables, and thus the number of final microstates will depend on the number of constraints and their scope, i.e. the number of elements influenced by the constraint. In the following, we consider examples with a different number and type of constraints, all resulting in the same number of microstates (four out of the eight viable states).

Table 1: Microstates of a three-bits system.

Microstate	
$\mu_1 = (1,1,1)$	$\mu_5 = (0,1,1)$
$\mu_2 = (1,0,0)$	$\mu_6 = (1,0,1)$
$\mu_3 = (0,1,0)$	$\mu_7 = (1,1,0)$
$\mu_4 = (0,0,1)$	$\mu_8 = (0,0,0)$

Table 2: Three bits microstates associated to the region of the phase space where an emergent property was observed $\{\mu_{\rm E}\}$, for a system with a single constraint of scope one.

$\{\mu_{\rm E}\}$
$\mu_1 = (1,1,1)$
$\mu_2 = (1,0,0)$
$\mu_6 = (1,0,1)$
$\mu_7 = (1,1,0)$

Hence, these constraints are codified in one bit of information, but we will see that the number of concepts needed to express these constraints can change from system to system.

The most simple macroscopic description associated to the observed ensemble, arises if we consider a coarse graining of the microscopic properties such that there is a surjective map between microstates and macrostates a macroscopic variable takes the value $'ON'$ if these microstates are visited and $'OFF'$ otherwise. In this way, only if there is a statistically significant bias towards these microstates we can say that a novel emergent macroscopic property is observed.

Taking these considerations in mind, we aim to disentangle the microscopic constraints in the system following our formalism. Given that we build our conceptual setting starting from the basic characteristics (obtained from measurements) and then performing binary logical operations, we expect that the results obtained for the different systems are fairly comparable. We will perform this comparison taking into account the number of propositions found and the concepts contained, i.e. analysing whether the respective representations provide a more or less compact description. Therefore, a valid description for the constraints should represent a reduction of dimensionality of the system (a compact description), as it is a necessary condition to build any simplified model.

System with a single constraint of scope one (S1). The rational is the same for the following three systems. We consider that there is an observed macroscopic emergent observation $\hat{c}_{\rm E}$, and we know the microstates associated to that particular region of the phase space $\{\mu_{\rm E}\}$. Then, we analyze the set of microstates looking for its constraints.

The first system we consider is a system where the first bit is constrained to a fixed value (c_1), leading to the observations $\{\mu_1, \mu_2, \mu_6, \mu_7\}$ that we explicitly show in table 2.

In order to find the system constraints we start comparing the concepts e_i, which determine the different microstates. We provide a compact representation of these comparisons with a network, see Fig. 2, where each concept e_i is linked with a concept e_j if they share a basic concept, c or d. Although the constraints determine the microstates, these act on the variables so we need to go one step further to identify them. We move from a network of microstates to a network of basic concepts, and we link two concepts c_i or d_i if they extend onto the same microstates (see Fig. 2). More formally, we link two concepts c_i and c_j with a directed edge if $\mathrm{Ext}(c_i) \subseteq \mathrm{Ext}(c_j)$, and with an undirected edge if $\mathrm{Ext}(c_i) \cap \mathrm{Ext}(c_j) \neq \emptyset$. In this way, we compactly represent all the dependencies present in the system represented with relationships of subordination (directed edges) cooccurrence (undirected) or exclusion (absent link) between the different values. This representation resembles the information that we would recover if we build a network of variables from a covariance matrix: positive (negative) correlation arises when similar (dissimilar) values are found between two objects

From the network relating variables in Fig. 2 it is easy to observe that one of the values of the first variable, d_1, is never observed, a fact that we can express with the proposition:

$$\mathrm{Ext}(d_1) = \emptyset$$

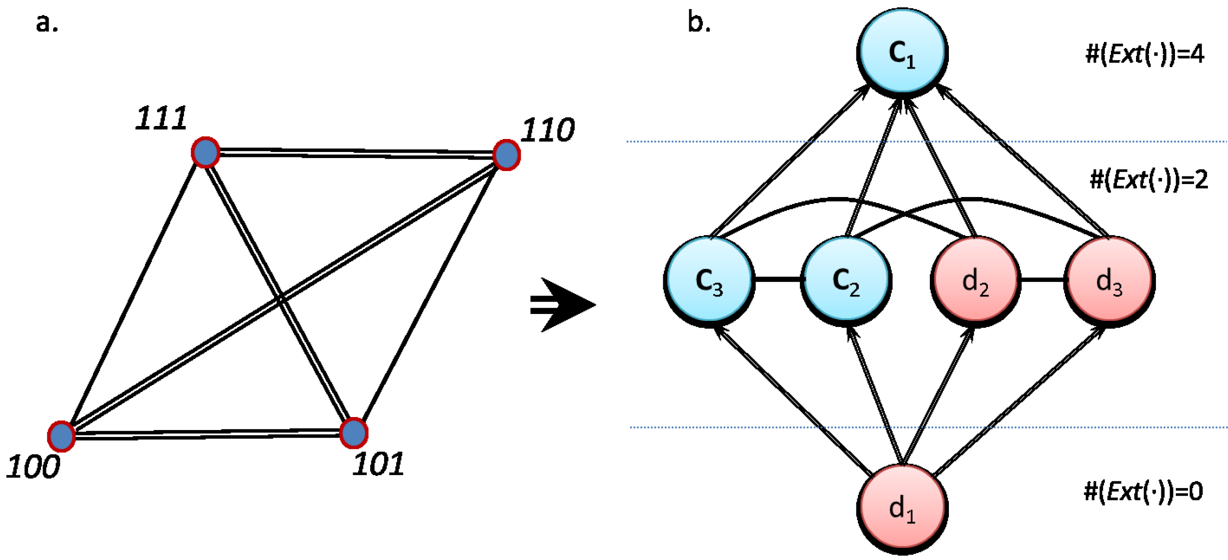

Figure 2: **Representations of a three bits system with a single constraint of scope one.** (Left) In the network, each node represents a microstate and it is linked with another microstate if they share the same observation for any component, where the number of links represent the number of concepts shared. (Right) Network of concepts extracted from the analysis of the microstates. Two links c_i and c_j are linked with a directed edge if $\text{Ext}(c_i) \subseteq \text{Ext}(c_j)$ and with an undirected link if $\text{Ext}(c_i) \cap \text{Ext}(c_j) \neq \emptyset$. The concepts are hierarchically ordered according to the cardinality of their extension, i.e. number of microstates they map. In this example a single constraint on x_1 naturally arises, as one of its possible values maps the empty set.

Table 3: Three bits microstates associated to the region of the phase space where an emergent property was observed $\{\mu_\text{E}\}$, for a system with two constraints of scope two.

$\{\mu_\text{E}\}$
$\mu_1 = (1,1,1)$
$\mu_4 = (0,0,1)$
$\mu_5 = (0,1,1)$
$\mu_8 = (0,0,0)$

What this proposition simply states is that, in order to identify that a given microstate belongs to this system, it is necessary to evaluate that the value measured on the first component of the system is different from zero. In other words, as soon as we reject the hypothesis H_0 : "d_1 is observed", we will be confident in that we are dealing with a microstate contained in $S1$. Note that we achieved a quite important reduction of the number of concepts needed to talk about the system: from the four concepts $\{e_1, e_2, e_6, e_7\}$, each of them containing three basic concepts, we reduce to a subset containing a single concept $U_\text{E} = \{c_1\}$. In addition, the set of non-emergent microstates are well described by the subset $U_\text{NE} = \{d_1\}$, what prevents the description for being vague.

System with two constraints of scope two (S2). We select now four microstates that are obtained imposing one constraint among each pair of variables. Taking the microstates $\{\mu_1, \mu_4, \mu_5, \mu_8\}$ (that are explicitly shown in table 3), and repeating the procedure of the previous example (see Fig. 3), we observe that the disconnected components in the graph lead to the following constraints, which can be expressed with the propositions:

$$\begin{aligned}\text{Ext}(c_1 \wedge d_2) = &\quad \emptyset\\ \text{Ext}(c_2 \wedge d_3) = &\quad \emptyset\\ \text{Ext}(c_1 \wedge d_3) = &\quad \emptyset\end{aligned}$$

It is easy to observe that one of these constraints is redundant. Given that c_2 and d_2 cannot be observed

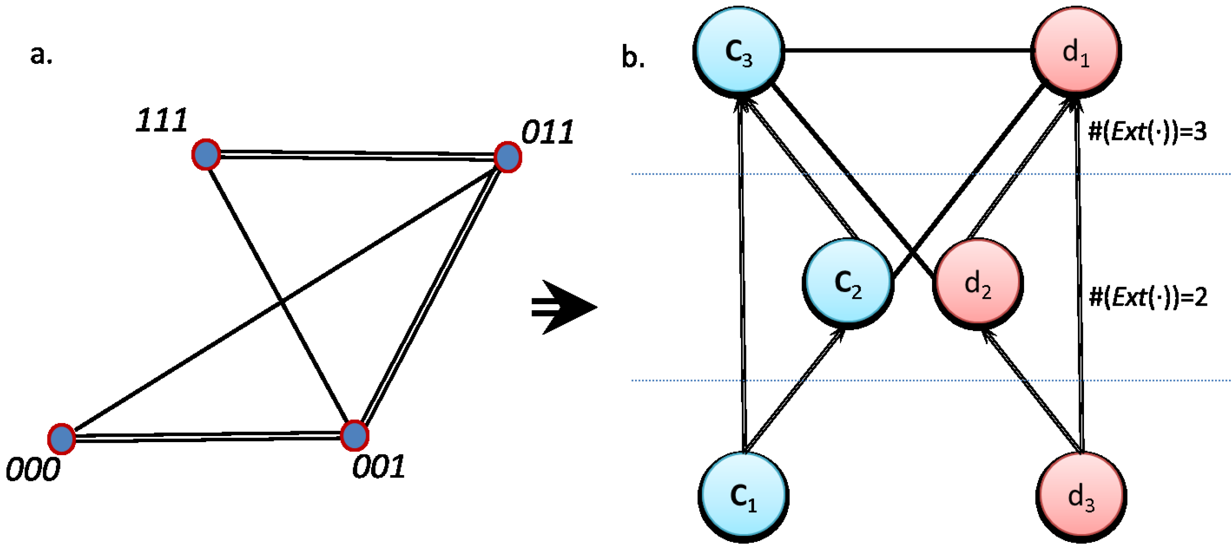

Figure 3: **Representations of a three bits system with two constraints of scope two.** (Left) In the network, each node represents a microstate and it is linked with another microstate if they share the same observation for any component, where the number of links represent the number of concepts shared. (Right) Network of concepts extracted from the analysis of the microstates. Two links c_i and c_j are linked with a directed edge if $\mathrm{Ext}(c_i) \subseteq \mathrm{Ext}(c_j)$ and with an undirected link if $\mathrm{Ext}(c_i) \cap \mathrm{Ext}(c_j) \neq \emptyset$. The concepts are hierarchically ordered according to the cardinality of their extension, i.e. number of microstates they map. In this example we identify the constraints observing those links that being viable are absent, for instance there is no link between d_3 and c_2.

Table 4: Three bits microstates associated to the region of the phase space where an emergent property was observed $\{\mu_\mathrm{E}\}$, for a system with a single constraint of scope three.

$\{\mu_\mathrm{E}\}$
$\mu_1 = (1,1,1)$
$\mu_2 = (1,0,0)$
$\mu_3 = (0,1,0)$
$\mu_4 = (0,0,1)$

simultaneously, if c_1 is observed it means that c_2 is also observed and thus d_3 cannot be observed. And the other way around, if d_3 is observed c_2 will not be observed and thus c_1 cannot be observed. Therefore, the third constraint $\mathrm{Ext}(c_1 \wedge d_3) = \emptyset$, is a consequence of the other two. We formulate our result positively saying that the set of emergent microstates $\{\mu_\mathrm{E}\}$ is described by the subset $U_\mathrm{E} = \{\{d_1 \wedge c_2\}, \{d_2 \wedge c_3\}\}$, and the set $\{\mu_\mathrm{NE}\}$ by $U_\mathrm{NE} = \{\{c_1 \wedge d_2\}, \{c_2 \wedge d_3\}\}$

System with a single constraint of scope three (S3; the parity bit system). Our last example is a set of microstates having an even number of *ON* bits, i.e. a single constraint involving all three components. This system has been previously introduced by Bar-Yam as a toy example of the particular type of emergent behaviour we introduced above called *strong emergence* [15]. For this system, given that we find two random values in two randomly selected bits, the third bit is constrained in such a way that the number of bits in the microstate is always odd. This rule is used in the control of message transmission, where the last bit (called the parity bit) is used to monitor the presence of errors in the chain transmitted. Note that we are not interested in understanding the system under this engineering perspective, as it provides already a rather *ad hoc* explanation on how the system is built [35] and, in this work, we assume no *a priori* knowledge of the underlying mechanisms generating the observation. It is just one possible observation that will be analysed as in the previous examples. The microstates we will consider are $\{\mu_1, \mu_2, \mu_3, \mu_4\}$, explicitly shown in table 4

In this case, the network of concepts intuitively resembles an sphere in the sense that there are no “borders”

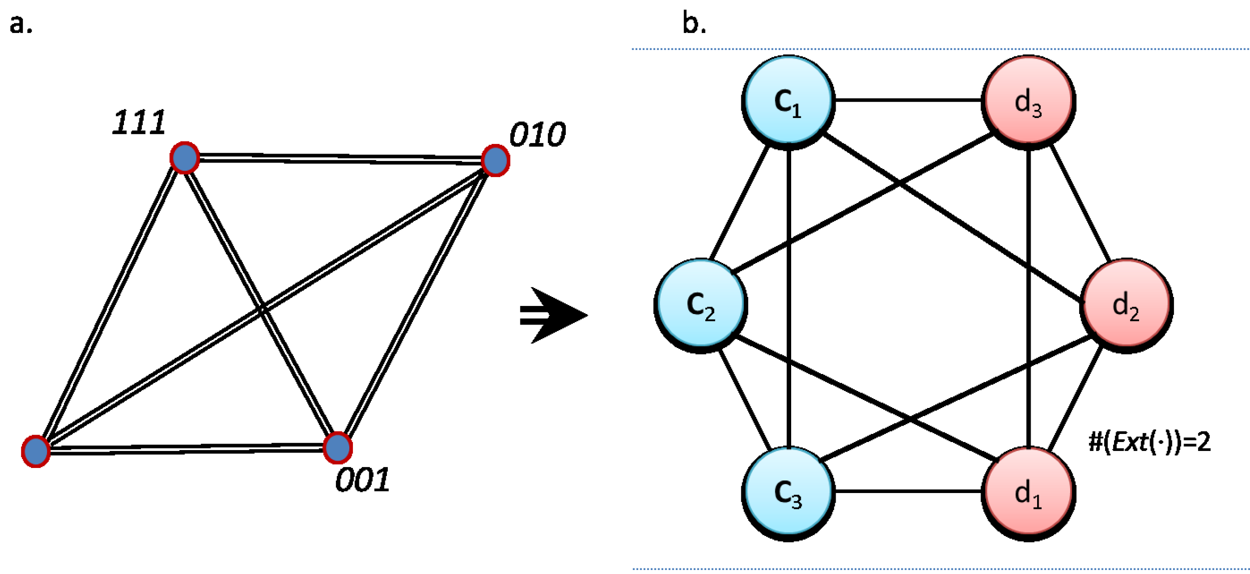

Figure 4: **Representations of a three bits system with one constraint of scope three.** (Left) In the network, each node represents a microstate and it is linked with another microstate if they share the same observation for any component, where the number of links represent the number of concepts shared. (Right) Network of concepts extracted from the analysis of the microstates. Two links c_i and c_j are linked with a directed edge if $\mathrm{Ext}(c_i) \subseteq \mathrm{Ext}(c_j)$ and with an undirected link if $\mathrm{Ext}(c_i) \cap \mathrm{Ext}(c_j) \neq \emptyset$. The graph of concepts is equivalent to the graph we would obtain for a free system, being just observed a reduction in the number of objects mapped by each concept (from $\#\mathrm{Ext}(\cdot)$=4 towards $\#\mathrm{Ext}(\cdot)$=2). It reflects the notion that the system has "no borders", and no constraints can be explicitly extracted with our framework.

–i.e. disconnected concepts from which propositions about the constraints are simply derived (see figure 4)–. Thus, the identification of constraints is possible only because we already know the viable values. Indeed, a parallel analysis of the free system highlights a lower cooccurrence of the different variable values, but there will be no differences in the final network topology we obtain. This fact would be also observed in the set of marginal probability distributions, as no bias will be observed for the free system nor for the parity bit system.

The comparison with the free system bring to the surface the following propositions:

$$\begin{aligned} \mathrm{Ext}(d_1 \wedge c_2 \wedge c_3) = \quad & \emptyset \\ \mathrm{Ext}(c_1 \wedge d_2 \wedge c_3) = \quad & \emptyset \\ \mathrm{Ext}(c_1 \wedge c_2 \wedge d_3) = \quad & \emptyset \\ \mathrm{Ext}(d_1 \wedge d_2 \wedge d_3) = \quad & \emptyset \end{aligned}$$

And what we observe is that the most compact way to talk about this system within this formalism is to write down all the microstates that are *not* observed. Therefore, we obtain no reduction of dimensionality at all what will make difficult to build any model. In this way, vagueness in the macroscopic concept $\hat{c}_{\mathrm{E}}$ still holds, because microstates from $\{\mu_{\mathrm{NE}}\}$ are getting attracted by those $\{\mu_{\mathrm{E}}\}$. More technically, $\mathrm{Bd}(\mathrm{Ext}(\hat{c}_{\mathrm{E}}) = \{\mu_{\mathrm{E}}\}) = \{\mu_{\mathrm{NE}}\}$, again because we do not consider a valid solution for solving vagueness of $\hat{c}_{\mathrm{E}}$ simply describing it through the subset $U_{\mathrm{E}} = \{e_1, e_2, e_3, e_4\}$, because it is not a *compact description*.

5 Emergence

5.1 The three-bits system in other representations

In the previous examples, we showed how our formalism worked differently depending on the type of constraints present in the system. We found that it apparently fails in getting a description of the constraints if there is a single constraint with a scope equal to the system's size. Of course, we may think that our formalism is simply insufficient and that there may be other more sophisticated formalisms that would be able to express the constraints in S3, with a much lower amount of information that the one we need to describe it. For instance, there may be a function q such that, given a description of a microstate e_i, returns a new *type* of concepts $\tilde{c} = q(e_i)$ that are able to differentiate the microstates of S3 from the rest. Let's see this with an example. We can characterize the three examples above with their probability distributions:

$$\begin{aligned} P(\mu) &= \delta(x_1, 1)/2^{n+1} && (S1) \\ P(\mu) &= \delta(\delta(H_{12}+1,1), H_{23}+1)/2^{n+1} && (S2) \\ P(\mu) &= \delta(\mathrm{mod}_2(\textstyle\sum_i x_i), 1)/2^{n+1} && (S3) \end{aligned}$$

where x_i is the value of the bit i, n is the number of bits, δ(a,b) is the Kronecker's delta, $H_{ij} = H(x_i - x_j - 1)$ is the Heaviside function and $mod_2(\cdot)$ is the module two function. In principle, there is no reason to think that the probability distribution of S3 is more complex than those of S1 and S2. For saying that it should be any objective reason to say, for instance, that the use of a summation operator and a module function is more complex than applying two Heaviside functions.

A fairer comparison can be assessed through the algorithmic information complexity (AIC) [36], also known as Kolmogorov complexity. We approximate this measure programming three simple scripts (one for every system) that generate strings of size N containing these constraints. In the Appendix, we provide the pseudocode for these programs along with some technical details, and the scripts can be found in the Supplementary Material. Compiling this scripts lead to three binaries which, after compression, have the following ratios of compression 3.070:1Kb (S1) $>$ 3.011:1Kb (S2) $<$ 2.998:1Kb (S3). With these results, the constraint implemented in S3 requires more information to be codified than those in S2 and S1, suggesting that it is a more complex constraint. Nevertheless, there is no dramatic difference as to identify it as particularly relevant or complex. In addition, in the Appendix we discuss that there are some choices in the way we implemented this exercise that may lead to different results, such as the ability of the subject writing the code.

What we aim to point out with these examples is that, the fact that the formalism we are using here has established limits, is not a drawback but an advantage to clearly set the boundaries of the framework. For instance, a desirable property of a framework should be that, if we increase the system's size, our ability to describe the constraints of a larger system should change according with their number and scope. If we think in how these systems will increase the number of microstates when the number of components N increases, both S1 and S3 will increase as 2^{N-1} while S2 increases as $N+1$. If we use AIC to quantify the constraints, the same script can generate a small or large string just changing the value of the variable N. According with our formalism, the number of propositions needed to describe the constraints in the system remains equal to one for S1, increases as $N-1$ for S2, and as 2^{N-1} for S3. We believe that this finding is remarkable and, in the following sections, we show which are the consequences of this observation, and how can be used to provide a quantification of emergence compatible with the scientific method.

5.2 Coverage excess

It is difficult to investigate the effects that an increase on system's size may have on traceability, because it would mean that we already have a full knowledge of the underlying constraints and how they would extend through the new elements, which is not typically the case when investigating complex systems. An alternative would be to *intervene* in the system neglecting components of the model and then monitor which is the relative change in our ability to predict the system's behaviour after the intervention. This strategy has been highlighted as a basic ingredient to link computational modelling with the scientific method [37]. When we neglect any component, we will reduce our predictive power if we lose constraints and, thus, there will be more states of the system compatible with the remaining constraints. We can quantify which is the uncertainty we generate when we lose constraints, what means that we can relate the causal effects that a component has on the other components, according to the notion of Granger causality [18].

We illustrate the proposal in Fig. 5 considering the three-bits synthetic examples again. Starting from any of the systems, we neglect the components one by one, and we explore which are the two-bit states recovered. Depending on whether any constraint was lost or not, we are able to build from these states a number of three-bit states. We simply introduce back into the system the variable neglected allowing it to get any viable value. Exploring systematically all the variables we infer which is the influence of the underlying constraints. For instance, for S1, if we neglect the first component x_1 –where the constraint relies–, we obtain all possible states of a two-bits system containing components x_2 and x_3. This is because the unique constraint in the

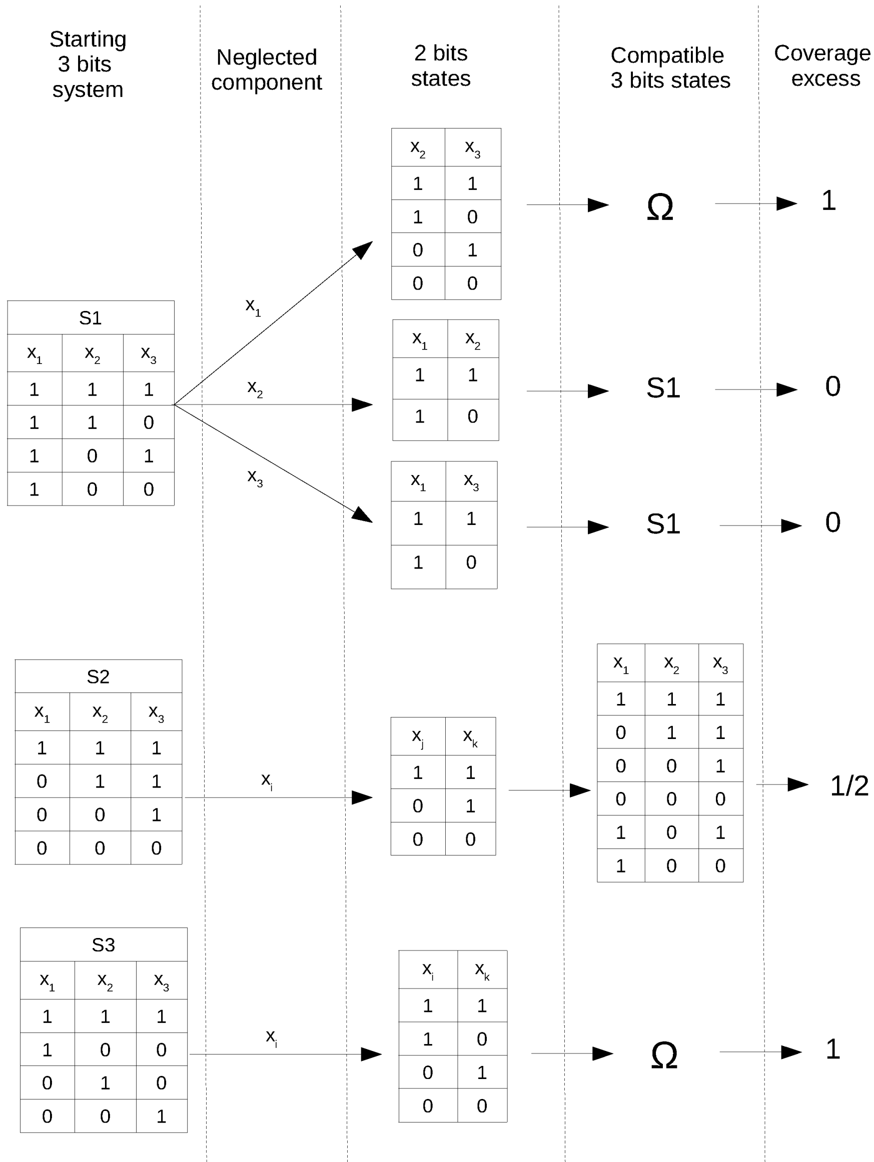

Figure 5: Scheme illustrating the definition of coverage excess. The scheme is divided in five columns (1-5) that we describe from left to right. (1) The three-bits systems under analysis in the main text are shown. If we intervene in the systems neglecting one component (2) we will obtain a set of 2-bits states (3). For the system S1, removing x_1 lead to different states than if x_2 or x_3 are removed, while S2 and S3 lead to the same states independently of the component removed (see Main Text for details). From the two-bits states, we recover the neglected component keeping it free of any constraint, which leads to a number of compatible 3-bits states (4). In the last column we show the result for the coverage excess obtained from this procedure using Eq. 8. The final value for the coverage excess of the system will be the average among the values obtained from the different interventions.

system was deleted, and thus we can recover the whole phase space Ω when we build up all the three-bit states compatible with these two-bit states. On the other hand, if we neglect any of the other two components x_2 or x_3, the constraint still remains in the first component x_1. Therefore, if we look into the three-bit states compatible with the two-bit states we are constrained for coming back to the original system S1. For the system S2, irrespective of the component neglected we reach the same two bits states. From them, we obtain not only the original three bits states but two more, what means that we have lost one constraint every time, but the other one is still active. For S3, irrespective of the component neglected, we obtain all possible two bits states, and hence the whole phase space will be always recovered, hence the constraint is always completely lost.

With this kind of interventions we can quantify how the original system is *covered in excess* when any intervention takes place. More formally, let's call U to the subset of concepts contained in the focus F describing a set of microstates $S = \{\mu\}$ containing N components in the phase space Ω, which we know is associated to a novel macroscopic property. Then remove any of the x_i components and consider a new set of microstates S' of a system with $N-1$ components. Next call $V(x_i)$ to the subset of concepts describing the set of microstates S'' of a system with N components obtained by adding a new unconstrained component x_i' to S'. We say that the coverage excess Ξ of S induced after neglecting the variable x_i and introducing the component x_i' is the quantity

$$\Xi(x_i) = \frac{\#(\mathrm{Ext}(V(x_i))) - \#(\mathrm{Ext}(U))}{\#(\mathrm{Ext}(F)) - \#(\mathrm{Ext}(U))} = \frac{\#(S'') - \#(S)}{\#(\Omega) - \#(S)} \tag{8}$$

where the function $\#(\cdot)$ returns the number of microstates contained in the set. This quantity takes a value of zero, when the states recovered are the same than the original ones, and it takes one when it is recovered the whole phase space. To consider a single value for the coverage excess we will average the result of the intervention over all variables:

$$\langle \Xi \rangle = \frac{1}{N} \sum_i \Xi(x_i) \tag{9}$$

If the system is very large an exhaustive computation may be unfeasible, and a random sampling of the different components or more complex interventions such as the removal of several variables, and we indicate this averaging over interventions with the brackets $\langle \cdot \rangle$. For the examples explored in the three bits system, we obtain $\langle \Xi \rangle_{S1} = 1/3$, $\langle \Xi \rangle_{S2} = 1/2$ and $\langle \Xi \rangle_{S3} = 1$. The coverage excess reflects the vulnerability of the system to this intervention and, as the effects have to do with the number of variables over which we intervene and with the scope of the constraints present in the system, it provides a mechanism to differentiate between upward and downward causation [38]. The system S1 is very vulnerable if x_1 is neglected but it is not affected at all if any other variable is neglected, and thus there is upward causation from the first variable to the whole system. On the other hand, the system S3 is very vulnerable, as the coverage excess is maximum irrespective of the variable over which we intervene, thus highlighting that there is a global constraint affecting the system downwards . Note as well that these values will scale differently with system size. While for $S1$ $\langle \Xi \rangle_{S1} \to 0$ when $N \to \infty$, for $S3$ it will remain constant and equal to one. Finally, we consider the particular case in which we are already dealing with a subset of microscopic concepts U such that an emergent process described by the macroscopic concept $\hat{c}$ is perfectly covered, i.e. such that $\mathrm{Ext}(U) = \mathrm{Ext}(\hat{c})$. In this case, we will say that the Eq. 9 provides *the coverage excess of the emergent property* $\hat{c}$, that we denote as $\left\langle \hat{\Xi} \right\rangle$.

5.3 Loss of traceability and emergence strength

The sensitivity of the knowledge of the observer when it intervenes on the system suggests that, in the research process, systems with higher coverage excess will be more difficult to analyse. This difficulty can be combined with the notion of traceability we proposed above. If we have a perfectly traceable system, we can quantify how much we lose traceability after intervention with the *loss of traceability*

$$\Upsilon = 1 - \left\langle \hat{\Xi} \right\rangle . \tag{10}$$

With this quantity we express that those systems which are easily covered in excess have a low traceability, and the other way around. Therefore, the associated macroscopic property will be difficult to explain, from which the following definition for the *emergence strength* σ of the emergent property arises naturally

$$\sigma = -\log(\Upsilon). \tag{11}$$

We took the natural logarithm of the traceability to represent the emergence strength as a distance between an state of perfect knowledge of the system (complete traceability) and the state after intervention. If, after intervention, we still remain in a situation of perfect traceability, this distance will be zero. On the other hand, if we completely lose all the information about the system in a way that, with the above procedure, we recover an unconstrained phase space, this distance will be infinity, thus reflecting some epistemological gap for these systems.

With these definitions we expect to reconcile different positions on whether the origin of emergence is epistemological or ontological: even if we deal with a perfectly traceable system –thus epistemologically accessible–, we can still see that there are systems that are particularly more inaccessible than others, and there are ontological reasons for that, namely the type of constraints involved in the system. For these systems, until perfect traceability is attained, we will probably be tempted to say that they are epistemological inaccessible, and that there is an strictly ontological and not epistemological reason for that. But it is a combination of both, there is an ontological reason why their emergence strength is so high as to build an epistemologically accessible (microscopic) description, but it does not mean that such description cannot be achieved at some point. We propose to solve controversies around the definition of weak and strong emergent properties [15] using the emergent strength: if the emergence strength is infinite we deal with a strongly emergent pattern whereas if the value is finite we deal with a weak emergent pattern with the associated strength as an indicator of how difficult it is to trace it.

Of course, we cannot discard that there exist systems epistemologically inaccessible. This may be the case for quantum, relativistic or computational systems –some of which the definitions of weak and strong emergence where originally thought–, but not for many systems of scientific interest (such as living systems) where we believe that the situation is rather the one that we expose here: these systems are very large and they are under constraints with a large scope. Note that it may be argued that, with the above definitions, it cannot be determined the emergence strength unless we achieved perfect traceability. This is only true for strong emergent properties because Eq. 10 can be easily modified to consider an intermediate state of knowledge, just using the expression in Eq. 9 (and not $\left\langle \hat{\Xi} \right\rangle$), which will give us an estimation of the emergence strength. We expect indeed that for natural systems there is a complex structure of constraints with different scopes, and we will be able to progressively discover this structure –possibly from low to high scopes–, and provide an estimation at any time.

6 Discussion

In this article we proposed a novel approach to investigate the concept of emergence in complex systems. We tackled the problem through the formalism of concrete topology, which is a constructive logical system that permits the investigation of the relationship between concepts and objects of observation [39]. In doing so, we focused in a particular kind of systems, which we believe attain much interest in the nowadays discussion of emergence. First, we considered that we are analysing a naturally occurring macroscopic emergent property, and not a process generated from a computation. In addition, we neglect any vitalism, what means nothing but accepting explanatory physicalism –using the words of Mitchell, what else could be?[2]–. To continue with, we considered that we are able to describe microstates of the system through experimental measurements. This implicitly assumes that we are able to differentiate the system from its background [23] and that we are able to provide a bottom-up characterization in terms of concepts associated to the elements that constitute the system [16]. Nevertheless, we considered as well that we have no clue on which mechanistic processes underlie these observations, as it happens when research on a new process is starting. When the underlying mechanism is well understood, there are already interesting proposals in the literature providing measures of emergence [40], but obviously this is not the case for most of the natural processes under research. Finally, we highlighted that there exist a relationship between the macroscopic observation of the emergent property and the constrained walk of the system in certain region of the phase space. With these conditions in mind, we aimed to talk about systems from which we expect to find sufficient regularities in the analysis of microstates as to be able to build explanatory models, i.e. in potentially robust emergent systems, which are the focus of the scientific interest [14].

Equipped with these tools and with a minimal description of complex system we showed that, when two descriptions coexist, the less detailed description is prone to generate dialectical concepts, that are vague unless a more explanatory description removes the ambiguity. This is the case between the macroscopic and microscopic description when an emergent property is observed, as it is typically not immediate to explain the emergent phenomena from the microscopic description. We showed that, building a microscopic model aimed to remove the ambiguity of the macroscopic observation, requires to identify constraints in the viable values of the microscopic variables. Interestingly, we identified concept disjunction as the basic logic operation to find constraints. Since long it has been recognized the importance of comparisons for the proper determination of any object, that may be viewed as a negative determination through the exploration of the limits of the object, as it was stated by Hegel [41]:

> "the object, like any determinate being in general, has the determinateness of its totality outside it in other objects, and these in turn have theirs outside them, and so on to infinity. The return–into–self of this progression to infinity must indeed likewise be assumed and represented as a totality, a world; but that world is nothing but the universality that is confined within itself by indeterminate individuality, that is, a universe."

With disjunction we explore the progression of an elementary object into other objects that may eventually lead to the identification of new objects exceeding the individuality of the starting objects, and then back: the identification of objects with a larger scope reinforce the individuality of the elementary objects. The search of similarity measures, dissimilarity measures or distances is an essential task in Biology and Ecology [42] aiming to understand, following a top-down approach, the information shared between the different observations. This is probably why methods comparing objects of observation, such as protein sequence alignments like BLAST [43], are the most cited ever in the scientific literature [44]. In general, disjunction is on the basis of dimensionality reduction techniques such as principal components analysis [45]. Following our framework, these are techniques aiming to obtain a representation with the minimum number of concepts explaining the maximum variability in the space of objects. In this way, we are able to *talk* about the set of objects using a subset of concepts, which is essentially the task addressed by dimensionality reduction techniques, and that we defined here with the notion of compact description.

We then applied these tools to three different ensembles of microstates of a three-bits synthetic system. We observed that the scope of the constraints is the main difficulty to identify them: larger is the scope of the constraint more difficult is to assess it. In particular, our method was unable to find a compact representation when the scope of the constraint has the same size than the system, which directly links the epistemological limitation of our framework with an ontological property of the system. We briefly visited other approximations, identifying that the type of constraints heavily influence the consequences that either an increase in system size or a loss of components may have in our ability to identify them. This observation seems to be independent of the formalism followed, and thus of any subjectivity induced by the one we chose here.

Notably, we were able to express this observation with concrete topology. We proposed a procedure based on the intervention of the observer on the system, thus compatible with the scientific method, to compute the loss of information experienced when we neglect components in the system. As this loss of information depends on the type of constraints present, we can quantify how difficult is to achieve traceability between the microscopic and macroscopic description. The loss of traceability was then used as a quantity to establish a distance between perfect traceability and our knowledge of the system after intervention, what we called the emergence strength.

We believe that, for the kind of systems we are analysing, the emergence strength paves the way to reconcile the differences between the notion of weak and strong emergence. In the systems we are interested, we aim to develop computational models to reproduce the experimentally measured data and simulate the emergent process, and thus it is compatible with weak emergence. Nevertheless, we propose to move the focus from the ability of building a computational model to simulate the process to an earlier scientific stage, namely the identification of constraints from experimental data. In the identification of constraints is where we start learning about the nature of the natural process we face. In this way, we focus in understanding the number and scope of the constraints, whose complexity will determine its emergence strength.

We conjecture that for systems with different types of constraints, those with smaller scope are identified first. Accordingly, if a system has only constraints with a large scope or there is a big gap with respect to those with lower scope, it may be simply impossible at a certain stage of knowledge to assess them, an example of this may be our nowadays knowledge of consciousness. For these processes, the emergence strength may be so large that it would be justified to call them strongly emergent processes. This definition seems compatible as well with the classification proposed by de Haan [8], as the existence of a microscopic emergent conjugated causally affecting the macroscopic pattern (in the strongest case, consciously), can be understood in terms of a global constraint (as he recognises relating this type of emergence with downward causation). This is the case in living systems where we believe strong emergence may be pervasive.

Natural selection can be viewed as a global constraint acting in very large temporal scales for every individual organisms. The constraints that one individual feels will have certain similarities with those felt by another individual (e.g. periodic exposition to day-light) being others specific to the individual's micro-environment. The term closure has been coined to described the fact that organisms and the environment (which includes other organisms) are entangled through interactions exerting a mutual influence such that natural selection is modulated by the activity of the organisms themselves [46]. This picture, in its stronger version (in which the influence is so high that the notion of individual as object of selection is challenged), becomes more and more important in nowadays research, overall in microbial world (see for instance [47]).

Consider the following example proposed in [48], in which we consider one individual for which its fitness f_i can be decomposed in two components, where the first component reflects the fitness f_{ij}^{int} of the individual as a consequence of its ecological interactions with other species j, and the second its fitness $f_i^{\overline{int}}$ due to any other process, thus $f_i = f_{ij}^{int} + f_i^{\overline{int}}$. Now consider a particular example, in which two individuals belong to two different species, a and b, interacting mutualistically. Finally, think in an evolutionary event which becomes fixed in the population of species a affecting its fitness, $f_a \to \hat{f}_a$, in such a way that the new fitness $\hat{f}_a < f_a$ and, in particular, $\hat{f}_a^{\overline{int}} = f_a^{\overline{int}}$ but $\hat{f}_{ab}^{int} < f_{ab}^{int}$. This means that the fitness of species b due to the interaction with species a will be also affected after the evolutionary event and, thus, there will be a change in the selection pressure of the regions of the genomes of both species codifying the traits needed for the interaction. Furthermore, if we consider an extreme scenario in which $f_{ab}^{int} \gg f_a^{\overline{int}}$ and $f_{ba}^{int} \gg f_b^{\overline{int}}$ –that may be the case for auxotrophs (see a synthetic ecological experiment in [49])–, the importance of these coevolving regions in the evolutionary process would be so high, that the conception of object of selection should be revisited [50]: it might be more appropriate to frame the evolution of both species considering them as a particular kind of multicellular species. This is probably why it has been emphasized the importance of mutualistic interactions in emergent processes [3], although recent theoretical results suggests that mutualism is not necessary to derive a mathematically precise definition of community-level fitness [51].

Not surprisingly, there is increasing interest in the development of methods for the inference of interactions, but the essence of the question remains the same. If we aim to understand emergent processes through interaction patterns, we will compare microstates representated through networks of interactions, and the main evolutionary constraints will be identified finding their common topological properties [48].

In summary, we find that the formalism we used here improves our ability to synthetically understand complex problems. We believe it could be used as well to face other challenging questions such as the concept of closure, and thus we hope that our effort will stimulate both the scientific and philosophical community. Looking for fresh formal approaches to talk about philosophical questions is particularly important because, just as formal frameworks help us in making predictions, they will help us as well in shaping the philosophical knowledge and to establish new links between science and philosophy. This would be probably good news for science, as the benefits of philosophy seem to be, for nowadays scientists, left behind.

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Appendix

In this appendix we provide the pseudocode to generate M strings of size N containing the constraints described in systems S1, S2 and S3 in the Main Text. An implementation in Fortran is available in Supplementary Material. The scripts have been developed trying to keep similar most of the structure for the three systems, which can be summarized as follows:

```
%Variables declaration
type x(N) % describes every bit in the system x={0,1}
for i in 1 to M

    for j in 1 to N
        if constraint
            x(j)=applyConstraint()
        else
            x(j)=rndGenerator()
        endif
        write x(j)
    endfor

endfor
```

Where the function `applyConstraint` applies the constraint correspondent to the condition `constraint`, which depends on the system, and the function `rndGenerator` randomly generates a zero or a one. For each system, the constraints are applied as follows:

```
% S1 constraint:
if(j==1)

    x(j)=1

endif
-------------------------
% S2 constraint:
if((x(j-2)==1)or(x(j-1)==1))

    x(j)=1

endif
-------------------------
% S3 constraint:
if(j==N)
    Remainder=MOD(Total,2)

    if(Remainder==0)THEN
        x(j)=1
    else
        x(j)=0
    endif

endif
```

where `Total` is a variable that sums-up all the $N-1$ previous values and `MOD` is a function that returns the remainder of a module two division of `Total`. Note that, while S1 and S3 have a straightforward generalization of their constraints from a 3-bits system to a N-bits system, it is not the case for S2 where several generalizations are possible. We decided to extend the system adding bits from left to right and keeping the original constraints along the chain. This will generate a bias in the distribution of the output microstates, rapidly decreasing the probability of generating strings containing many zeros for increasing N.

Next, the scripts have been compiled with gfortran (gcc versión 4.8.5 in Red Hat 4.8.5-4) and the binaries compressed with both bzip2 (version 1.0.6) and gzip (version 1.5), obtaining qualitatively similar results. It is important to remark that the scripts should be coded with as few dissimilarities as possible. For instance, S2 and S3 require two additional variables respect to S1, which already make a difference in the compression ratios. But, if for whatever reason, we decide to use for these two variables integer types for S2 and real types for S3 (or the other way around) the results may be qualitatively different. This is one of the reasons why the complexity of the system is difficult to assess with this kind of approximation. Although our implementation is close to be optimal, as the size of the binaries before compression is almost the same (13467Kb for S1 and S2 and 13519Kb for S3) we find that concrete topology provides a fairer framework for comparing different systems.

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