EXERCISES (ii) - Maths for Biology

Computational Methods in Ecology and Evolution
Imperial College London
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Lecturer: Alberto Pascual-García.

Correspondence: alberto.pascual.garcia@gmail.com (informal questions); apascual@ic.ac.uk (official submissions)


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Abbreviations:
 [EXM] This is a type of exercise that you may find in the exam.
 [DLV] This is an exercise that you must deliver before the exam.
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Exercises Limits

Exercise 1. Definition of limits [EXM].

1. Demonstrate, applying the definition of limit, that $\lim_{x\to 2}(x^2 + x + 2) = 8$. Clue: Just apply literally the definition, and factorize one of the conditions arising in the definition.

2. The floor and ceiling functions are widely used in computational modelling to round a given real number to its nearest lower ($\lfloor x \rfloor$) and upper integers ($\lceil x \rceil$), respectively. Demonstrate that $\lim_{x\to 2}\lceil x \rceil$ does not exist.

Exercise 2. Indeterminate limits of type $0/0$ [DLV]. In a recently published paper in which you are involved you found that, when the concentration of a given protein $[p_1]$ was in a range of 0.1 to 0.2 mg/mL, the concentration of another protein $[p_2]$ which is regulated for $[p_1]$, approximately follows the function $f([p_1]) = [p_2] = \exp([p_1])(1 - \exp(-[p_1]))$.

Two months later, the PI of your lab (who is the corresponding author) received a letter from the editor of the journal saying that another lab tried to reproduce these results with a different technique, and that they submitted a comment claiming that, for a range of p_1 between 0.2 and 0.4 mg/mL, the function $f([p_1])$ strongly overestimates the observed behaviour, which is better fitted by the function $g([p_1]) = [p_2] = -(1+[p_1])\ln(1-[p_1])$.

The PI of your lab replied to the editor that, for realistic cell conditions, the concentrations of $[p_1]$ are much lower than 0.1 mg/mL and, therefore, the second lab is reporting values that are far from being realistic while the experimental conditions you used should be a better approximation. Therefore, in the biologically relevant region, you expect your fit to be better. Nevertheless, the referees considering both the comment and the PI reply said that the experimental ranges of the experiments in your lab are not realistic either, and accepted the comment of the other lab that was finally published. Now, you can reply to this comment and you decide to proceed as follows to understand what's going on:

- Plot both functions in your computer between a value very close to zero and 0.4 and check if these guys may be right when they say that you are overestimating the actual value of $[p_2]$ for the values they explored.

- Observe what is happening for very low concentrations.
- Demonstrate that, in the regime of low concentrations (when $[p_1] \approx 0$) both results may be correct but, instead of doing this demonstration computationally (you could look for a procedure similar to the one you used in the exercise "approximating hyperbolic functions") you will show, using the L'Hopital rule, that both functions are infinitesimal equivalents, i.e. you will demonstrate that:

$$\lim_{x \to 0} \frac{f([p_1])}{g([p_1])} = 1.$$

Exercise 3. Indeterminate limits of type 1^∞ [EXM]. In extracellular electrophysiology experiments made in your lab, you were able to show that the response of a population of neurons $f(V)$ is negatively related with the magnitude of an externally applied voltage $V > 0$, and that it is well approximated by the function $f(V) = \cos(V)^{1/\sin(V)}$. You submitted the paper, and one referee says that, although your function is very cool, it is not defined for $V = 0$, and this is a biologically relevant value because it is the resting potential of the population when the stimulus is absent. You will show to the referee that, even if the function is not defined for $V = 0$, the $\lim_{V \to 0} f(V)$ exists and corresponds with the expected resting value ($f(0) = 1$).

Clue: You will need to consider the relationship for indeterminate limits of type 1^∞ and then infinitesimal equivalences found in the survival toolbox. To deal with these type of indeterminate limits, we proceed as follows: when we have a limit of the type $\lim_{x \to a} f(x)^{g(x)}$ we can always use the following transformation $e^{\lim_{x \to a} g(x) \log f(x)} = e^{\lim_{x \to a} g(x)(f(x)-1)}$. The last step is explained because $\log(f(x))$and $f(x)-1$ are infinitesimal equivalents when $f(x) \to 1$:

$$\lim_{f(x) \to 1} \frac{\log(f(x))}{f(x)-1} = \lim_{f(x) \to 1} \frac{f'(x)/f(x)}{f(x)-1} = \lim_{f(x) \to 1} \frac{1}{f(x)} = 1$$

This is similar to what it was shown in the exercise of logarithmic derivatives.

Exercise 4. The Central Limit Theorem [DLV] Go to the article of Wikipedia talking about the Central Limit Theorem and read the main definitions. Then try to explain with your own words (as simple as you can and without equations), the difference between the "Classical" Central Limit Theorem and the Lyapunov Central Limit Theorem. In particular, I would like you to try to understand this equation:

$$\lim_{n \to \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} E\left[|X_i - \mu_i|^{2+\delta}\right] = 0$$

And you will probably need to understand first what a moment means. Finally, have a quick look to this paper [1] and try to read at least until the paragraph talking about Erwin Shrödinger. You may consider to read in Wikipedia about Taylor's law. But, more importantly, if you haven't read the book of Schrödinger... do it!

Exercises Derivatives

Exercise 1. Mean value theorem [EXM]. In a blog, one scientist affirms that she has recorded the highest speed ever quantified for a herd of certain mammal escaping from a volcano eruption. She said that, at the moment of the eruption at time t_0, she was recording them in position x_0 but that, 10 minutes later she was in a safe area at point $x_1 = x_0 + 2.5$km and the herd just appeared. Then, she reasoned that, using the mean value theorem, it is possible to demonstrate that these animals should be moving at least at one instant of time faster than the maximum speed ever recorded (12 km/h). You should show if she is right or not.

Exercise 2. Taylor expansion [EXM]. Obtain the order two expansion of the function$f(x) = \frac{\ln x}{x}$ at $x_0 = 1$.

Exercise 3. Simple derivatives [EXM]. Compute the derivatives of the following functions, providing a simplified expression:

$$f(x) = \arctan\left(\frac{\sin x}{1+\cos x}\right)$$
$$g(x) = \log\left(\frac{2\tan x + 1}{\tan x + 2}\right)$$

Exercise 4. Functions of the type $f(x)^{g(x)}$ [EXM]. When we find functions of the type $f(x)^{g(x)}$, we can use again a trick to find the derivative using logarithms. Consider the following function:

$$f(x) = (1+x)^{\log(1+x)}$$

If we take logarithms in both sides we obtain:

$$\log f(x) = \log(1+x)\log(1+x)$$

Now, we perform the derivatives of both sides:

$$\frac{f'(x)}{f(x)} = \frac{1}{1+x}\log(1+x) + \frac{1}{1+x}\log(1+x) = \frac{2}{1+x}\log(1+x)$$

And then we solve for $f'(x)$:

$$f'(x) = \frac{2\log(1+x)}{1+x}(1+x)^{\log(1+x)}.$$

Now, follow a similar reasoning to compute the derivative of:

$$g(x) = (a^2+x^2)^{\arctan(x/a)}.$$

Exercise 5. Logistic or Verhulst curve [DLV] A classical model for the growth of a population is given by the logistic equation:

$$N(t) = \frac{KN_0}{N_0 + (K-N_0)e^{-rt}}$$

Evaluate both $N(t=0)$ and $lim_{t\to\infty}N(t)$ and explain the meaning of the constants N_0, K (both are positive). Plot the function with your computer for different values of the constants. Then, compute the first derivative, and interpret the situation in which it would be zero. Repeat the analysis with the second derivative. Help: When computing the derivatives of functions, you should remember that expressions like products of constants work as a single constant. To operate, you may consider to transform the function like this: Call $A = KN_0$, and $B = (K - N_0)$, and then note that:

$$N(t) = \frac{A}{N_0 + Be^{-rt}} = \frac{Ae^{rt}}{N_0e^{rt} + B}.$$

To interpret the meaning of the root of the second derivative, you may consider to check the sign of the second derivative in a neighborhood of the point in which it is zero.

Maximum and minimum

Exercise 6. River management [DLV]. The courses of two rivers R1 and R2 follow the equations $y - x^2 = 0$ and $x - y - 2 = 0$, respectively. The water flow of R2 has heavily decreased due to human activity, and the government decided to tackle the problem regulating these activities and building a channel between both rivers (of course, not extent of controversy). You belong to the consultancy committee evaluating the ecological impact, finding that this impact strongly depends on the length of the channel. As a consequence, you should determine which are the two closest points of the rivers where the channel should be built. Furthermore, politicians are very happy with you because a minimal channel implies a minimal cost. Given that every kilometer costs around 1,5 million pounds, you should also determine the final cost of the channel. Clues: Plot in your computer both equations. Determine the equation of a straight line going from an arbitrary point of one of the rivers to the other and take derivatives.

Exercise 7. Functions behaviour [EXM]. Determine the points where the following functions have relative maxima and minima:

$$\begin{aligned} f(x) &= xe^x \\ g(x) &= \log\sqrt{2x^3+3x^2} \end{aligned}$$

References

[1] Wayne S Kendal and Bent Jørgensen. Taylor's power law and fluctuation scaling explained by a central-limit-like convergence. *Physical Review E*, 83(6):066115, 2011.