EXERCISES (v) - Maths for Biology

Computational Methods in Ecology and Evolution
Imperial College London
Silwood Park

Course 2017-18

Lecturer: Alberto Pascual-García.

Correspondence: alberto.pascual.garcia@gmail.com

Exercises linear algebra

Unless otherwise is stated, all vectors and matrices are given in the standard basis.

Basic Operations

Exercise 1. Systems of equations [EXM] SON LOS MISMOS QUE DI EN CLASE, CAMBIAR
Solve the following systems of equations:

1.

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x + 5y - 8z &= 4 \\ 3x + 8y - 13z &= 7 \end{aligned}$$

2.

$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

3.

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y - 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

Clue: Each of the systems corresponds to one of the situations described by this Theorem: Any system of linear equations has either i) a unique solution, ii) no solution, iii) infinity solutions.

Exercise 2. Distributive law of vectors Consider the following vectors a, b, c and the scalar k:

$$a = (1, 2, 3)$$
$$b = (4, 0, 1)$$
$$c = (3, 2, 2)$$

$$k = 2$$

and demonstrate that:

1. $a(b + c)^t = ab^t + ac^t$ (where the superscript t stands for the transpose).
2. $(a + b)c^t = a^t c + b^t c$
3. $a(kb^t) = (ka)b^t + k(ab^t)$

Exercise 3. Matrix basics (i) Consider the following matrices A, B, C, and the scalar k:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 5 & 1 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 3 & 4 \\ 9 & 6 & 9 \end{pmatrix};$$

$$k = 3$$

and test if the following expressions hold:

1. $C = A + B = B + A$
2. $C^t = A^t + B^t = B^t + A^t$
3. $A + (B + C) = A + B + C = (A + B) + C$
4. $A(kB^t) = (kA)B^t$
5. $AB^t = BA^t$

Exercise 4. Matrix basics (ii) Consider the following matrix B and the vectors a and b:

$$a = (1, 1, 2); \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}; \quad c^t = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix};$$

And compute:

1. $D = aB$
2. $E = aBc^t$
3. $F = aBa^t$

Exercise 5. Matrix basics (iii) Consider the following matrices:

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 1 & 2 \end{pmatrix} \quad ;$$

And compute:

1. $C = AB$
2. $D = BA$

Exercise 6. Matrix basics (iv) Consider the following matrices:

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 2 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \; M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And compute:

1. $C = D_1 A$
2. $D = AD_1$
3. $E = D_2 A D_1$
4. $F = D_1^2$
5. $G = AF$
6. $H = IA = AI$

Exercise 7. The contact overlap and the designability [DLV] Reproduce the Table 1 and the Figure 6 of the Snake Puzzle paper (Nido *et al.* in the folder Readings/Teaching in Blackboard), providing the scripts.

Basic Properties

Exercise 1. Trace [EXM] Consider the following THEOREM: Let's assume that $A = (a_{ij})$ and $B = (b_{ij})$ are squared matrices and k a scalar. Then:

1. $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.
2. $\text{tr}(kA) = k\text{tr}(A)$
3. $\text{tr}(AB) = \text{tr}(BA)$

In this problem I just want you to get used on mathematical notation, and I ask you to demonstrate that this theorem is true. I show you how the first point is demonstrated. Let me call $A + B = C$. The idea when we demonstrate properties of matrices is that we change the notation to an explicit representation in which we "see" what happens with the cells values, we operate on cells values (which are scalars and we know well how to operate with them) and then we come back to matrix notation:

$$\text{tr}(A + B) = \text{tr}(C) = \sum_{k=1}^{n} c_{kk} = \sum_{k=1}^{n} (a_{kk} + b_{kk}) = \sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = \text{tr}(A) + \text{tr}(B).$$

Now you should demonstrate points 2 and 3 proceeding similarly.

Exercise 2. More complex expressions with matrices [EXM] We can use equations in more complex expressions. Consider for example the function $g(x) = x^2 - x - 8$, where x is a square matrix. It may sound weird, because we have the number 8 and we haven't talked about the sum of a matrix and scalar. Therefore, if we see this kind of expression we should consider that the independent term is a number multiplying an identity matrix with the same size than x. Taking this into account, demonstrate that $A = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}$ is a root of $g(x)$, i.e. show that $g(A)$ is equal to the null matrix.

Exercise 3. Determinants (i) Consider a generic matrix B obtained after applying elemental operations on the matrix:

$$A = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$$

and check if the following theorem holds. Note that I am not asking you to proof the theorem, because to proof the theorem I would ask you to demonstrate that it holds for *any* operation, and I just ask you to check if for *one* operation of your choice it is true:

THEOREM:

- If you interchange two rows or columns in A, then $|B| = -|A|$. (Obtain the matrix B doing the operation mentioned and test the statement.)
- If you multiply any row or column by a scalar k,then $|B| = k|A|$.
- If you sum a multiple of a row or column to another row (column) (e.g. $R_2 = R_2 + kR_1$, where R_i is the row i and k is an integer), then $|B| = |A|$.

Exercise 4. Determinants (ii) [DLV] When we have a square matrix of dimension $n > 3$ we said that it is not so easy to compute the determinant, and that we should compute it using its minors. If the matrix is very large it might be a mess, so ideally we would like to perform operations in the matrix until we get just a single minor of size $n = 3$. In addition, it is useful to keep in mind the following

THEOREM: Let's consider A a square matrix

- If A has any rows (or column) filled of zeros, then $|A| = 0$.
- If A has two rows (or columns) that are equal, then $|A| = 0$.
- If A is triangular, what means that A has only zeros above or below the diagonal then $|A|$ is equal to the product of the diagonal elements only. For instance, for the identity matrix I the determinant will be $|I| = 1$.

Let's see an example. Compute the determinant of the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{pmatrix}$$

I can consider the third result shown in the theorem of Exercise 3, and do the following elemental operations. I will get the row number two R_2 and I will use it to modify the other rows as follows: $R_1 \to R_1 - 2R_2$ (I subtract two times the second row to the first one). Then, $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 + R_2$. Note that we always used the second row, that we will call the "pivot" row (or column). With these operations I obtain the matrix:

$$A' = \begin{pmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}$$

And I know, by means of the theorem of Exercise 3, that $|A'| = |A|$. If you realize, the third column has all the elements zero except the element $a'_{23} = 1$. Therefore, to compute the determinant, if I consider that column, there is only one minor "surviving" (because all the others would be multiplied by zero). Then, computing the determinant of A' is equivalent to compute the determinant of the matrix:

$$A'' = \begin{pmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

You can easily proof that $|A''| = 38$. Now you should follow the same reasoning to compute the determinant of the scary matrix:

$$B = \begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$$

Exercise 5. Inverse of a matrix Look for the inverse of the matrices:

$$A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}; \; B = \begin{pmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{pmatrix}$$

and remember this THEOREM: If A is a square matrix, it is equivalent to say:

- A has an inverse (A^{-1}).
- $AX = 0$ has only the trivial solution.
- The determinant of is different from zero: $|A| \neq 0$.

Some Geometry

Exercise 1. Geometry Given the vectors $a = (1, 2)$ and $b = (0, 2)$ calculate:

- The length of the vectors.
- Their scalar product.
- The projection of a on b.
- The angle between both vectors.

Exercise 2. Gram-Schmidt orthonormalization [EXM]

Consider the basis given by the vectors $a_1 = (2, 1, 2)$, $a_2 = (3, -1, 5)$ and $a_3 = (0, 1, -1)$ and transform it into an orthonormal basis using the Gram-Schmidt procedure.

Exercise 3. Other geometric operations [DLV] If you plot the following four points in the plane, you will see that they represent a square:

$$P = (1, 1); \quad Q = (1, -1); \quad R = (-1, 1); \quad S = (-1, -1);$$

Now consider the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and apply each of these matrices to all four vectors and, for each matrix, plot the four new points it generates. You can do this transformations with the computer if you want, but you should keep in mind that to observe the effects it is necessary that you are able to distinguish which is each point before and after the transformation, i.e. you should keep the points labeled at all times. Then, determine which matrix performs one of the following operations: i) a reflection, ii) a central dilation, iii) a stretch, iv) a shear and v) a permutation.

Basis and diagonalization

Exercise 1. Cayley-Hamilton [EXM] In a previous exercise, we observed that the matrix $A = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}$ was a root of $g(x)$. Now, the Cayley-Hamilton THEOREM states that every matrix is a root of its own characteristic polynomial. Demonstrate that this is true with this matrix as well.

Exercise 2. Basis [EXM] Determine if the following vectors constitute a basis in $\mathbb{R}^3$:

$$a = (1, 1, 1);\ b = (1, 2, 3);\ c = (2, -1, 1)$$

Exercise 3. Subspaces [EXM] Two subespaces of $\mathbb{R}^3$ that we will call U and W are described by the following vectors:

$$u_1 = (1, 1, -1);\ u_2 = (2, 3, -1);\ u_3 = (3, 1, -5);$$

$$w_1 = (1, -1, -3);\ w_2 = (3, -2, -8);\ w_3 = (2, 1, -3).$$

You should represent each subespace with a matrix and, reducing each matrix to its row canonical form, show that $U = W$.

Exercise 4. Changing the basis [EXM] Consider the following basis in $\mathbb{R}^3$:

$$S = \{u_1 = (1, 2, 0), u_2 = (2, 3, 1), u_3 = (0, 2, 3)\}$$

You should look for:

- The matrix P that changes the basis from the standard basis $E = \{e_1, e_2, e_3\}$ to the basis of S.
- Take a generic vector (a, b, c) and write it in the basis of S.
- Which is the matrix Q that allows you to change back the coordinates of the basis S to E?
- Bring the coordinates of generic vector you obtained in the basis S back to the basis E.

Exercise 4. Diagonalization (i) [EXM] Diagonalize the following matrix A, and find the matrix P such that $D = P^{-1}AP$, being D the diagonal matrix. Once you obtain D and P, demonstrate that $A = PDP^{-1}$. Finally, find the value of A^5.

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$$

Exercise 5. Diagonalization (ii) [EXM] Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$$

1. Diagonalize the matrix A, and find the matrix P such that $D = P^{-1}AP$, being D diagonal.
2. Find the matrix P^{-1}.
3. Verify the Cayley-Hamilton theorem, which states that any square matrix is a root of its characteristic polynomial.

Exercise 6. Diagonalization (iii) [EXM] Look for the eigenvalues of the following matrix B, and then look for a set of eigenvectors linearly independent. Is it B diagonalizable?

$$B = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$